Francis Gregor
Exeter Coll
Oxfor

Francis Gregory
Exeter Coll:
Oxford

[illegible]
[illegible]
[illegible]

ELEMENTS

OF

CONIC SECTIONS

DEDUCED FROM THE CONE,

AND DESIGNED AS

AN INTRODUCTION

TO THE

NEWTONIAN PHILOSOPHY.

BY THE

REV. A. ROBERTSON, D.D. F.R.S.

SAVILIAN PROFESSOR OF ASTRONOMY IN THE UNIVERSITY OF OXFORD.

OXFORD,

AT THE CLARENDON PRESS.

MDCCCXVIII.

TO THE REVEREND

CYRIL JACKSON, D.D. F.R.S.

LATE DEAN OF CHRIST CHURCH,

EQUALLY EMINENT FOR HIS OWN ABILITIES AND LEARNING,

AND

FOR HIS UNIFORM ENCOURAGEMENT AND PROMOTION

OF TALENTS AND ACQUIREMENTS IN OTHERS,

AS A

TESTIMONY OF THE HIGHEST ESTEEM FOR HIS CHARACTER,

AND AS A

TRIBUTE OF GRATITUDE FOR MANY IMPORTANT FAVOURS,

THIS WORK

IS MOST RESPECTFULLY INSCRIBED,

BY HIS MUCH OBLIGED AND FAITHFUL SERVANT,

THE AUTHOR.

ADVERTISEMENT.

THE following Work is founded on one which the Author published in the year 1802, when he was Savilian Professor of Geometry; but it differs from that Treatise * in so many places, both in the nature and form of the demonstrations, that he deemed it more proper to announce this as a new publication, than an abridgment of the former.

As in the former publication, so in the present, there are two points of which it seems necessary that the reader should be apprized; first, what previous knowledge will be expected from him; and secondly, what extent of information the Treatise itself is intended to afford. The first of these will prevent the young stu-

* The following is the title of the work alluded to: "A "Geometrical Treatise of Conic Sections, in four Books. "To which is added, a Treatise on the Primary Properties "of Conchoids, the Cissoid, the Quadratrix, Cycloids, the "Logarithmic Curve, and the Logarithmic, Archimedean, "and Hyperbolic Spirals."

dent from entering upon the work till he is duly prepared, and the second will enable him to judge how far it is likely to contribute to the attainments which he has in view.

It is expected then, that the young student should understand thoroughly the first six Books of Euclid, the first twenty-one Propositions of the eleventh Book, the two first of the twelfth, and the first principles of Algebra.

As no number can be assigned as a limit to the properties of the conic sections, any treatise on the subject can be supposed to contain only a selection of those which are most important and most useful, either generally, or with reference to the particular design of the Writer. In the present instance the design has been, to furnish the young Mathematician with such a series of propositions as might prepare him for considering some of the most important truths in science, and enable him to enter on the study of natural philosophy, with the prospect of obtaining a thorough knowledge of the subject. According to these views the selection of properties and the extent of the work have been regulated; and at the same time the arrangement and division of the whole have been made with a design of accommodating two descriptions of readers. Those who are considered as constituting the first class are supposed to be desirous of a general but respectable portion of knowledge of the subject. For the use of such, a perusal of the three Books will be found sufficient, as they contain the properties of the sections most frequently referred to in pure and mixed mathematics. For those who rank under the second, or higher description, a knowledge of the whole will be requisite, as it completes the original design of rendering the work a preparative for the Newtonian Philosophy. The Author flatters himself indeed, that he shall be found to have carried his elucidations of the Principia, in the following sheets, considerably beyond what is usual in treatises of Conic Sections.

Something must now be added concerning the particular method, which has been adopted in these sheets, of deducing the primary properties of the sections from the nature of the cone.

It is well known, that about the middle of the seventeenth century a difference of opinion took place among mathematicians concerning the proper source from which the properties of the conic sections should be deduced. But notwithstanding the objections which then began to be made to their deduction from the cone, and which have since been continued, it appears to the Author of this work, that the difficulties attributed to the deductions from it were not to to be imputed to the solid itself, but that they were occasioned solely by the manner in which the deductions had been made. The early writers did not happen to perceive that the general and extensive property, expressed in the forty-eighth article of this Treatise, could easily be obtained from the cone; and not adverting to this, their deductions from the cone were sometimes tedious and intricate.

The above-mentioned property, as far as secants are concerned, occurs (I believe for the first time) in a folio volume, of which a treatise of conic sections makes a part, entitled, "Euclides Adauctus et Methodicus," &c. published by Guarinus in 1671. The property to the same extent is to be found in Jones's "Synopsis Palmariorum Matheseos," published in 1706; but neither of these two authors considered the property as a fundamental one, nor do they seem to have been aware of the advantages it was capable of producing. Its extensive utility was first evinced in Hamilton's Conic Sections published in Latin in 1758; and on the appearance of this work objections to the cone ought to have ceased.

This was my persuasion when I published my first Treatise * on the subject; and every deli-

* In the year 1792 the author of the present work pub-

beration on the subject since has tended to strengthen my conviction of its justice for the following reasons. First, the whole trouble with the cone is reduced to a very few demonstrations, for which no farther knowledge of Euclid is necessary than what is requisite for Spherical Trigonometry. Secondly, by this method the general properties are obtained with most ease and elegance. Lastly, by deducing the properties from the cone the treatise is rendered more extensively useful. No work on conic sections, confined to their description on a plane, can be applied to elucidations in Perspective, Projections of the Sphere, the Doctrine of Eclipses, and in some other particulars of the highest importance in science.

For the rest it need only be said, that the manner, in which the properties of the sections are classed and arranged, appeared to the Author, on the whole, to be that which was best calculated to shew what properties are general, and what are appropriate to each of the sections.

N. B. When a demonstration, in the follow-

lished a quarto volume, entitled, "Sectionum Conicarum Libri septem. Accedit Tractatus de Sectionibus Conicis, et de Scriptoribus qui earum doctrinam tradiderunt." The last mentioned Tract contains a full historical account of the subject.

ing work, is effected by means of two ranks of magnitudes, which, taken two and two in the same or in a cross order *, have the same ratio to one another, they are placed thus,

$$\begin{array}{c} A : B : C : D \\ E : F : G : H; \end{array}$$

A, B, C, D representing the first rank, and E, F, G, H the second. Previous to this arrangement of the magnitudes, their ratio to one another is established, and therefore it evidently appears in which of the two orders the magnitudes are proportional. In order to arrange the magnitudes in this manner, it was necessary in a few places to use this mode of contraction,

$$\left.\begin{array}{l} t.\ AB^2 \\ \quad or \\ s.\ AB^r. \end{array}\right\}$$, which means the square of A B if a tangent, or the rectangle under its segments if a secant.

Besides the common signs of addition, subtraction, equality, and multiplication, > and < are used in some articles. The sign > may be read *greater than*, for, being put between two quantities, it is used to indicate that the quantity which precedes is greater than what follows. The sign < may be read *less than*, for, being

* That is either ex æquali, or ex æquali in proportione perturbata.

put between two quantities, it is used to indicate that the quantity which precedes it is less than what follows.

When an article is designated only by a number, it is to be understood as a corollary to the definition, proposition, or lemma immediately preceding it.

ELEMENTS

OF

CONIC SECTIONS.

BOOK I.

Containing general Properties of the Sections deduced from the Cone.

Fig. 1.

ART. 1. IF through the point V, without the plane of the circle A F B, a straight line A V D be drawn, and extended indefinitely both ways, and if the point V remain fixed, and the straight line A V D be moved round the whole circumference of the circle, two Superficies will be generated by its motion, each of which is called a *Conical Superficies;* and these mentioned together are called *Opposite Superficies.*

2. A straight line drawn from the fixed point V to any point G in either superficies is wholly in that superficies; and, being produced, the part on the other side of V is wholly in the opposite superficies. For a straight line having the same position which the generating line A V D had, when it passed over the point G, is in the superficies; and, being produced, the part beyond V is in the opposite superficies. Hence the article is evident; for only one straight line can be drawn from V to G, as two straight lines cannot inclose a space.

BOOK I.

3. The solid contained by the conical superficies and the circle A F B is called a *Cone*.

4. The fixed point V is called the *Vertex of the Cone*.

5. The circle A F B is called the *Base of the Cone*.

6. Any straight line drawn through the vertex of the cone to the circumference of the base is called a *Side of the Cone*.

7. A straight line V C, drawn through the vertex of the cone and the center of the base, is called the *Axis of the Cone*.

8. If the axis of the cone be perpendicular to the base, it is called a *Right Cone*.

9. If the axis of the cone be not perpendicular to the base, it is called a *Scalene Cone*.

10. A plane is said to *touch a conical superficies*, when it meets the superficies, and being produced indefinitely, in any direction, falls without the superficies.

11. A straight line which meets a conical superficies, and which, being produced both ways, falls without the superficies, is called a *Tangent*; but a straight line which meets a conical superficies in two points, or each of the opposite superficies in one point, is called a *Secant*.

12. A straight line is said to be parallel to a plane, when both being produced indefinitely both ways, they do not meet.

13. If a cone be cut by a plane, their common intersection is called a *Conic Section*.

14. The common intersection of any plane, not passing through the vertex of the cone, with the conical superficies, is called the *Curve of a Conic Section*.

PROP. I. Fig. 1. 15. If a cone (V A F B) be cut by a plane passing through the vertex (V), the section will be a triangle.

For let the plane, passing through the vertex V, cut the plane of the base A F B in the straight line (3. xi.) A B, and the circumference of the base in the points A, B; and let the straight lines V A, V B be drawn. Then, as the points V, A, B are in the plane cutting the cone, the straight lines V A, V B are wholly in the same plane; and as the points A, B are in the conical superficies, the straight lines V A, V B are also wholly in the superficies, by article 2. The straight lines V A, V B are therefore the common intersections of the conical superficies and the plane cutting the cone; and consequently the section V A B is a triangle. BOOK I.

16. If a plane, passing through the vertex, cut a cone, it will cut the opposite superficies in two straight lines, and only in those two. For if the plane V A B be extended on both sides of the vertex, it will cut the opposite superficies in the straight lines V A, V B produced, and in them only. This is evident from the last, and art. 2. Fig. 1.

17. If either of the opposite conical superficies be cut by a plane parallel to (A B D) the base of the cone, the common intersection (F G H) of the superficies and the plane will be the circumference of a circle, and its center will be in the axis of the cone. PROP. II. Fig. 2.

For let C be the center of the base, and let V C, the axis of the cone, cut the plane F G H in the point I. From the point I, and in the plane F G H, draw any two straight lines I F, I G to the conical superficies. Through V I C, I F let a plane be passed, and let it cut the superficies in the side V F A, and the base of the cone in the straight line C A. Let a plane also be passed through V I C, I G, and let it cut the superficies in the side V G B, and the base of the cone in the straight line

BOOK I. C B. Then (16. xi.) F I, A C, and also G I, B C are parallel to one another, each to each; and (29. i.) the triangles A C V, F I V, and also the triangles B C V, G I V are equiangular, each to each.

Consequently (4. vi.) A C : F I :: V C : V I,
and V C : V I :: B C : G I;
and therefore (11. v.) A C : F I :: B C : G I.
But A C=B C, and therefore (14. v.) F I=G I.

In the same manner it may be proved that any other straight line drawn in the plane F G H from the point I to the conical superficies is equal to F I; and therefore F G H is the circumference of a circle, and the point I, in the axis V C, is its center.

18. From the last, and art. 1. it appears that any circle parallel to the base of the cone, having its center in the axis, and its circumference in either of the opposite superficies, may be taken for the base of the cone.

19. The solid contained by the conical superficies V F G H, opposite to V A B D, and the circle F G H, is a cone; and the cones V A B D, V F G H, having the common vertex V, and whose superficies are opposite, being generated by the same line, as in art. 1, are called *Opposite Cones.*

Fig. 3. 20. If the plane V B E touch the conical superficies in the side V B, and the cone V A B F be cut by the plane F D C parallel to the plane V B E, the section F D C, formed by the cutting plane and the cone, is called a *Parabola.*

21. The plane V B E is called the *Vertical Plane to the Parabola.*

22. As the cone may be indefinitely extended, it is evident that the parabola may also be indefinitely extended; and as the parabola does not surround the cone, it is evident that its curve does not include a space.

23. An indefinite number of straight lines parallel to VB may be drawn in the plane of the parabola. For if a plane pass through VB and any point in the parabola, its common section with the plane of the parabola (16. xi.) will be parallel to VB. BOOK I.

24. If the cone VABC be cut by a plane, and the section DKLH, formed by the plane and the cone, surround the cone, and is not a circle, it is called an *Ellipse*. Fig. 4.

25. If the opposite cones VABE, VMN, be cut by a plane VBE passing through the vertex V, and be also cut by a plane parallel to VBE, forming with the opposite cones the sections FDC, QRS; each of the sections FDC, QRS is called an *Hyperbola*, and when mentioned together they are called *Opposite Hyperbolas*. Fig. 5.

26. The plane VBE is called the *Vertical Plane to the Hyperbola, or Opposite Hyperbolas*.

27. It is evident, as each of the opposite cones may be indefinitely extended, that an hyperbola may be indefinitely extended, and that its curve does not include a space.

28. An indefinite number of straight lines parallel to VB or VE may be drawn in the plane of the opposite hyperbolas. For if a plane pass through VB, or VE, and any point in either of the hyperbolas, its common section with the plane of the hyperbolas will be parallel (16. xi.) to VB or VE.

29. A straight line in the plane of a conic section, which meets the curve, and which being produced both ways falls without it, is called a *Tangent*; but a straight line which meets the curve of a conic section in two points, or each of the opposite hyperbolas in one, is called a *Secant*.

30. *Scholium*. Although only the Parabola, Ellipse,

BOOK I. and Hyperbola, are denominated Conic Sections, the attentive reader will readily perceive from the foregoing articles, that five different Sections may be formed by the intersections of a cone and a plane varying its position. For if a straight line parallel to the base be within the cone and remain fixed, and a plane move about it as an axis, when the plane passes through the vertex, the intersection of the cone and plane will be a triangle, as in article 15. When the plane has moved from the vertex, but still cuts both the opposite cones, the section formed in each will be an hyperbola, as in article 25. When the plane, proceeding in its motion round the fixed straight line, has arrived at a position parallel to that of a plane touching the cone in one of its sides, the section which it then forms with the cone is a parabola, as in article 20. In any other position of the moving plane, besides those already mentioned, an ellipse or circle will be formed with the cone.

PROP. III. Fig. 6. 31. If a straight line (D E) touch a conic section (G D H), a plane passing through (V) the vertex of the cone and the tangent (D E) will touch the cone (V A C B) in that side (V D C) which passes through the point of contact.

For, as D E touches the section in D, every point in D E, excepting D, falls without the curve of the section, and consequently without the cone. It is therefore evident, from article 2, that any straight line drawn from V, excepting V D C, in the plane passing through V D C, D E, will fall without the conical superficies, and that this plane will touch the cone in the side V D C.

PROP. IV. 32. If a straight line (D E) touch a conic section (G D H), no other straight line (I D) can touch it in the same point (D).

For, if possible, let I D also touch the section in D, and through V and I D let a plane pass and cut the plane of the base of the cone in the straight line K C. Then as, by the last article, the plane V C K touches the cone in the side V C, the straight line K C touches the circle in C. For the same reasons if the plane passing through V and D E cut the plane of the base in the straight line F C, this line will also touch the circle in C. The two straight lines therefore K C, F C are tangents to the circle at C, which (16. iii.) is impossible. BOOK I.

33. If a straight line as D E touch the conical superficies, or a conic section G D H, and a side V D C of the cone be drawn through D the point of contact, a plane passing through this side of the cone and the tangent D E will touch the superficies of the cone; and being produced beyond V, it will touch the opposite superficies in V C produced. This is evident from the last two articles.

34. If the section G D H be an hyperbola, the tangent D E cannot meet the opposite hyperbola. For D E is the common intersection of the plane V C F and the plane of the section G D H, and, by article 31, the plane V C F touches the opposite superficies in C V produced. It is therefore evident from article 25, that the tangent D E cannot meet the opposite hyperbola.

35. *Lemma.* If two straight lines (A D, B C) be parallel, and if planes (A D F, B C F) pass through them and cut one another (in E F), the common section (E F) will be parallel to each of the two first mentioned straight lines. Fig. 7.

For let A D be at right angles to a plane A B E, and then (8. xi.) B C is also at right angles to the plane A B E. Consequently each of the planes A E F D, B E F C (18. xi.) is at right angles to the plane A B E, BOOK I. and therefore E F (19. xi.) is at right angles to the same plane, and E F (6. xi.) is parallel to A D, B C.

PROP. V. Fig. 14, 15. 36. If a ſtraight line (V G) pass through the vertex (V) and fall without the opposite cones (V A M B, D V E), two planes, and only two, can be drawn through it to touch the superficies, and these planes will be on the opposite sides of a plane (V C F) passing through the straight line (V G), and cutting the base (in C F).

Fig. 14. First let the straight lines V G, C F be parallel. Let C F be bisected in L, and draw I L M at right angles to C F, and let it meet the circumference of the circle in the points M, I. Let a plane pass through the straight line V G and the point I, and this plane will touch the superficies. For let it cut the plane of the base in the straight line I K. Then as C F, V G are parallel, and as the plane of the base passes through C F, and the plane V I K passes through V G, the intersection I K of these planes will be parallel to C F, by article 35. But as I M bisects C F at right angles, it passes through the center of the circle (Cor. 1. iii.), and as I K, C F are parallel, and as the angle I L F is a right one, the angle K I L is also a right one (29. i.), and therefore I K (16. iii.) touches the circle in I. Consequently, by article 33, the plane passing through V I, I K, or, as above, through V G, V I, touches the superficies; and in the same manner it may be proved that the plane passing through V G and the point M touches the superficies.

Fig. 15. Secondly, let the straight line V G be not parallel to C F, but let it meet it in K. Draw K I, K M (17. iii.) touching the base in I, M; and then the plane passing through V G, I K, and also that passing through V G and M K, will touch the superficies, by article 33.

It is evident, in either case, that no other plane, besides the above-mentioned two, can pass through V G, and touch the superficies; and that one of these tangent planes is on the one side, and the other on the opposite side of the plane passing through V G, C F. BOOK I.

37. If a conic section surround the cone, two straight lines, and only two, parallel to one another, can be drawn to touch the section; if the section does not surround the cone, no straight line, parallel to a tangent, can be drawn to touch the section; but if the section be an hyperbola, one straight line, and one only, parallel to a tangent, can be drawn to touch the opposite hyperbola. PROP. VI.

Part I. Let the section D H L K surround the cone V A F B, and let the straight line G D touch the section in the point D: another straight line, and only one, parallel to G D, can be drawn to touch the section. Fig. 4.

For let V D A be the side of the cone passing through D, the point of contact, and let a plane pass through V A, D G, and this plane will touch the conical superficies in the side V A, by article 33. In this plane, and through V, the vertex of the cone, draw V T parallel to D G. Then V T will fall without the opposite cones; and by article 36, another plane can be passed through V T touching the conical superficies. Let this plane touch the superficies in the side V L B, and let its intersection with the plane of the section D H L K be L I. Then as L I is in the plane touching the cone in the side V L B, it meets the conical superficies in the point L only. It will therefore meet the curve of the section D H L K in the point L only, and consequently it will touch the section: and as the plane T V L I passes through V T, and the plane of the section passes through D G pa-

BOOK I. rallel to V T, by article 35, L I is parallel to D G. And as no other plane passing through V T can touch the conical superficies, besides the two T V A, T V B, it is evident, from article 35, that no other straight line besides L I, parallel to D G, can be drawn to touch the section D H L K.

Fig. 3. and 5. Part II. Let F D C be a section which does not surround the cone, and let D G touch the section in the point D. No other straight line, parallel to D G, can be drawn to touch the section.

For let V B E be the vertical plane to the parabola, or hyperbola, as in articles 20, 21, 25, 26. Let V D A be the side of the cone passing through D, the point of contact; and through V D A, D G let a plane pass, and let it cut the vertical plane in the straight line V T. Then, by article 33, the plane V D G will touch the conical superficies, and (16. xi.) D G, V T will be parallel; and as V T is in the plane, touching the conical superficies in the side V D A, it will fall without the opposite cones. Another plane, therefore, and only one, can be passed through V T to touch the conical superficies, by article 36. But when the section is a parabola, the other plane passing through V T, and touching the superficies, is the vertical plane V B E, which is parallel to the parabola. When the section is an hyperbola, then the vertical plane V B E passes through V T, and cuts the base of the cone in the straight line B E; and supposing T V L to be the other plane passing through V T, and touching the conical superficies, the planes T V L, V D G are on opposite sides of V B E, by article 36. Consequently the plane T V L cannot meet the hyperbola F D C. It therefore follows from the above, and article 36, that if the section does not surround the cone, no straight line, parallel to a tangent, can be drawn to touch the section.

Part III. Let F D C, Q R S be opposite hyperbolas, and let D G touch the hyperbola F D C in the point D. Then one straight line, and only one, parallel to D G can be drawn to touch the opposite hyperbola Q R S. BOOK I. Fig. 5.

For, every thing remaining as in the preceding part, through the sides V A, V L, in which the planes T V A, T V L, passing through T V parallel to D G, touch the superficies, let a plane be passed, cutting the vertical plane in the straight line V W and the plane of the hyperbolas in the straight line D R. Then as V W, R D, L V are in the same plane, and as (16. xi.) V W, R D are parallel, and L V meets V W, it will (ax. 12. i.) also meet R D. Let them meet in the point R. Then as the plane T V L touches the opposite cone M V N in L V produced, it will meet the plane of the hyperbolas in the point R. Let the intersection of these two planes therefore be R X; and as the plane of the hyperbolas passes through D G, and the plane T V L passes through T V parallel to D G, by article 35, R X is parallel to D G; and being in the plane touching the conical superficies, it will touch the hyperbola Q R S in the point R. It is also evident, for the same reasons as are mentioned above, that no other straight line parallel to R X, or D G, can be drawn to touch the hyperbola Q R S.

38. If a straight line touch a conic section, a straight line drawn through any point within the section, and parallel to it, will meet the curve in two points. For let every thing remain as in the last article, and let P be any point within the section. Through V T and the point P let a plane be passed, and let this plane cut the plane of the section in the straight line H K; and, by article 35, H K will be parallel to G D the tangent, and also to V T. Now as the point P is within Fig. 3, 4, 5.

BOOK I. the section it is also within the cone, and therefore, by article 15, the plane passing through V T and the point P will cut the cone in two sides; and as these two sides and V T, H K are in the same plane, and V T, H K are parallel, H K will meet each of these two sides, by (ax. 12. i.); and as H K is in the plane of the section, it must meet the curve of the section in the same points in which it meets these two sides of the cone. It is also evident from the above, and article 37, that a straight line drawn through any point within the opposite hyperbola Q R S, and parallel to G D, will meet the curve Q R S in two points.

39. If a straight line meet the curve of a conic section in two points, two straight lines may be drawn parallel to it to touch the section, if it surround the cone; but if the section does not surround the cone, only one straight line parallel to a secant can be drawn to touch the section; and, if the section be an hyperbola, only one straight line parallel to a secant can be drawn to touch the opposite hyperbola. For, let H, K be the points in which the secant H K meets the curve of the section; and through H K and V, the vertex of the cone, let a plane be passed, and, if the section surround the cone, draw V T in this plane parallel
Fig. 4. to H K. Then as this plane, by article 16, can only cut the oppofite superficies in straight lines drawn through V the vertex and the points H, K, it is evident, that V T must fall without the opposite cones. Consequently, by article 36, two planes can be passed through V T to touch the conical superficies, one on each side of H K; and the intersections of these planes with the plane of the section will touch the section, and, by article 35, these tangents will be parallel to
Fig. 3. and 5. H K. If the section does not surround the cone, let the plane passing through the secant H K and V

cut the vertical plane V B E in the straight line V T, and, for the same reasons as are mentioned above, V T will fall without the opposite cones. Then through V T two planes may be passed, by article 36, touching the conical superficies. But, according to article 38, only one of these planes can meet the parabola, and one of them can meet the hyperbola, and the other the opposite hyperbola; and, by article 35, G D, the intersection of the plane T V A with the plane of the section F D C, will be parallel to H K, and R X the intersection of the plane T V L with the plane of the opposite hyperbolas, will also be parallel to H K, and each of the straight lines G D, R X must touch the section which it meets. BOOK I.

40. If a straight line meet the curve of a conic section in two points, any straight line parallel to it, drawn through a point within the same section, or, if the section be an hyperbola, within the opposite hyperbola, will also meet the curve of the section, in which it is drawn, in two points.

For, by article 39, a straight line parallel to the first mentioned straight line can be drawn to touch the section or opposite hyperbola; and, by article 38, a straight line drawn through any point within the section and parallel to the tangent will meet the section or opposite hyperbola in two points. Hence (30. i.) this article is evident.

41. If a straight line (C D) meet each of the curves (C A, D B) of two opposite hyperbolas in one point (C, D), a straight line (A B) parallel to it, drawn through any point in the plane of these sections, will also meet each of the curves of these opposite hyperbolas in one point. PROP. VII. Fig. 8.

For let V be the vertex of the opposite cones, in which the hyperbolas are formed, and through C D and

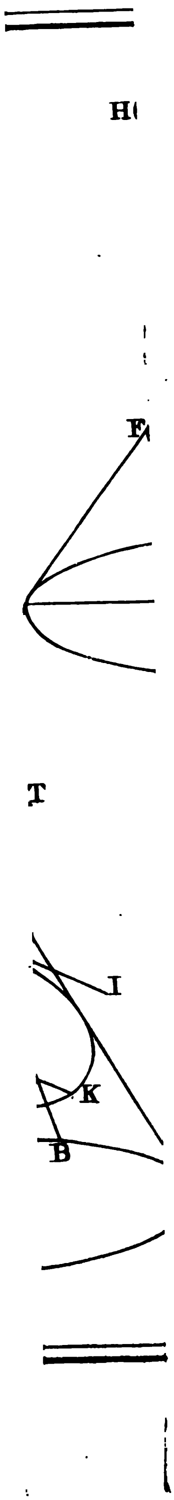
BOOK I.

V let a plane be passed, and let it cut a plane passing through A B and V in the straight line V T. Then the plane passing through C D and V must cut the opposite cones in the sides V C, V D; and, by article 35, V T is parallel to C D and also to A B. Again, as V T is in the plane V C D, and as this plane, by article 16, cuts the opposite superficies only in V C, V D, or in these lines produced, it follows that V T falls within the opposite cones. The plane passing through VT, A B will therefore cut the opposite cones in two sides; and from the above, (and ax. 12. i.) A B will meet one of these two sides in the one superficies, and the other in the opposite superficies. But as A B is in the plane of the opposite hyperbolas A C, B D, it must meet the curve of each hyperbola and the superficies, in which this hyperbola is, in the same point. The straight line A B will therefore meet the curve of each of the opposite hyperbolas in one point.

42. If a straight line meet each of the curves of opposite hyperbolas in one point, it will fall wholly without the hyperbolas, but being produced it will fall on the one side within one hyperbola, and on the other within the opposite hyperbola. For it is evident that C D is without the opposite cones, and that being produced it falls on the one side within one cone, and on the other side within the opposite cone.

Fig. 9. 43. *Lemma.* If two straight lines (A B, C B) cutting one another be parallel to a plane (D G H E), a plane passing through them will be parallel to the same plane.

For let F be any point in the plane D G H E. Through A B and F let a plane be passed, and let it cut the plane D G H E in the straight line D F H; and let a plane passing through C B and F cut it in E F G.


H
F
T
I
K
B

Then will A B be parallel to D F H, and C B will be parallel to E F G. For if not, then A B will meet D F H, and C B will meet E F G, and consequently A B, C B will meet the plane D G H E, in which D H, E G are, contrary to the hypothesis. The plane passing through A B, C B (15. xi.) is therefore parallel to the plane D G H E. BOOK I.

44. *Lemma.* If A B, B C, D E, E F, G B, B H, I E, E K be eight straight lines, and first Fig. 10.

if the hypothesis be $\begin{cases} AB : BC :: DE : EF, \\ GB : BH :: IE : EK; \end{cases}$

then $AB \times GB : BC \times BH :: DE \times IE : EF \times EK$.

Or, secondly, if the hypothesis be

$AB \times GB : BC \times BH :: DE \times IE : EF \times EK$,

and $AB : BC :: DE : EF$;

then $GB : BH :: IE : EK$.

In order to prove the preceding assertions, let the first and second, the third and fourth, the fifth and sixth, the seventh and eighth, of the eight straight lines, taken two and two, be in the same straight line; and let these straight lines be at right angles to one another, and let the rectangles be completed as represented in the figure. Then by the first hypothesis (and 1. vi.) and placing for ex æquali,

$AB \times GB : BC \times GB : BC \times BH$

$DE \times IE : EF \times IE : EF \times EK$, and therefore

$AB \times GB : BC \times BH :: DE \times IE : EF \times EK$,

which is the first assertion.

Again, by the second hypothesis and inversion (and 1. vi.) and placing for ex æquali,

$BC \times GB : AB \times GB : BC \times BH$

$EF \times IE : DE \times IE : EF \times EK$, and therefore

$BC \times GB : BC \times BH :: EF \times IE : EF \times EK$.

Consequently (1. vi.), $GB : BH :: IE : EK$.

45. If a straight line (G H) touching either, or cut- PROP. VIII.

BOOK I. ting one or both of the opposite superficies, meet a tangent (F P) or secant (D E) parallel to the base of

Fig. 11, 12, 16. the cone, its square, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the tangent, or the rectangle under the segments of the secant, which it meets, in the same ratio, wherever the point of concourse may be in the first mentioned line*.

Fig. 11, 12. Let V L be drawn through the vertex parallel to G H, and first let it meet the plane of the base in L, and let the plane passing through the parallels V L, G H cut the base in L B A, and the cones in the sides B V K, A V I. Through D E let a plane be passed parallel to the base, and let its line of common section with the plane of the triangle A V B be I K. Let the secant G H meet the conical superficies, or opposite superficies in G, H, and let it meet D E and I K in F. Then as G H is parallel to V L, and (16. xi.) I K to A L, the triangles V B L, H K F are similar, as are also the triangles V A L, G I F, and

$$VL : AL :: GF : IF, \text{ and}$$
$$VL : LB :: FH : FK.$$

Consequently, by article 44,

$$VL^2 : AL \times LB :: GF \times FH : IF \times FK.$$

But, by article 17, I D K E is a circle, and therefore (35. or Cor. 36. iii.) $IF \times FK = DF \times FE$. If, when the point F is without the circle, F P be drawn touching the circle in P, then (36. iii.) $DF \times FE = FP^2$; and therefore $VL^2 : AL \times LB :: GF \times FH : DF \times FE$ or FP^2, the secant D E or the tangent F P being parallel to the base.

* When two secants meet one another, the segments of either of the two are its parts between the point of concourse and the points in which it meets the superficies; and if a tangent meet a secant, or another tangent, its magnitude is limited by the point of concourse, and its point of contact.

If G H touch the superficies in T and meet D E in the plane of the circle D K P E in F, let the plane passing through the parallels G H, V L touch the base in C, the point in which V T meets the base, and cut the plane of the circle D K P E in F P, and then F P will touch the circle in P. Then as V L is parallel to T F and (16. xi.) L C to F P, we have V L : L C :: F T : F P, and therefore, (22. vi.) $VL^2 : LC^2$ or $AL \times LB :: FT^2 : FP^2$ or $DF \times FE$. BOOK I. Fig. 16.

If V L, and consequently G H, be parallel to the base, the section formed with the cone by a plane passing through G H and D E or F P will be a circle by article 17. In this case therefore (36. iii.) the square of G H, if a tangent, or the rectangle under its segments, if a secant, will be to the square of F P, a tangent, or the rectangle under the segments of D E, a secant, in the ratio of equality, wherever the point of concourse may be in G H.

46. Every thing remaining as in the last article, if a second straight line be parallel to G H, and touch either or cut one or both of the opposite superficies, and also meet a straight line parallel to the base and which touches or cuts either superficies, the square of this second, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the line which it meets, if a tangent, or the rectangle under its segments, if a secant, either as VL^2 to $AL \times LB$, or the ratio will be that of equality.

For if this second straight line, and consequently G H, be not parallel to the base, by article 35, a plane passing through it and V will cut the plane V A B L in the straight line V L, and therefore, in this case, by the last article, (and 35. and 36. iii.) the preceding assertion is evidently true.

If this second straight line, and consequently G H,

BOOK I.

be parallel to the base, then the section formed with the cone and a plane passing through the second and the straight line which it meets, will be a circle, by article 17; and in this case, as in the last of article 45, the ratio above mentioned is that of equality.

47. *Scholium.* As every point in the curve of a conic section is also in the conical superficies, it is evident that all the articles demonstrated concerning straight lines touching or cutting the conical superficies, or opposite superficies, may be transferred to straight lines, which in the same manner touch or cut a conic section, or opposite hyperbolas.

PROP. IX. Fig. 17.

48. If there be four straight lines in the plane of a conic section, and if A B the first meet C B the second, and D E the third meet F E the fourth, and if the first be parallel to the third, and the second to the fourth, and if each of them either touch or cut a conic section, or cut opposite hyperbolas; then the square of A B, if a tangent, or the rectangle under its segments, if a secant, will be to the square of C B, if a tangent, or the rectangle under its segments, if a secant, as the square of D E, if a tangent, or the rectangle under its segments, if a secant, to the square of F E, if a tangent, or the rectangle under its segments, if a secant.

Case 1. If the straight lines A B, C B, and consequently D E, F E, be each parallel to the base of the cone in which the section was formed, or the base of the opposite cone, the section must be a circle, by article 17, and the ratio above stated will be that of equality.

Case 2. Let A B, D E be not parallel to the base of the cone, in which, or in which and its opposite, the section or opposite hyperbolas were formed; but let C B, F E be parallel to the base, and then, by article 46, (and 11. v.) the assertion is evidently true.

BOOK I.

Case 3. Let neither A B nor C B, and consequently neither D E nor F E, be parallel to the base of the cone; but suppose B G, E H to be straight lines parallel to the base of the cone in which the section, or in which and the opposite cone the opposite hyperbolas were formed; and let B G, E H touch or cut either of the opposite conical superficies. Then by article 46, (and 11. v.) the square of A B, if a tangent, or the rectangle under its segments, if a secant, will be to the square of B G, if a tangent, or the rectangle under its segments, if a secant, as the square of D E, if a tangent, or the rectangle under its segments, if a secant, to the square of E H, if a tangent, or the rectangle under its segments, if a secant. Again, by the same and inversion, the square of B G, if a tangent, or the rectangle under its segments, if a secant, is to the square of C B, if a tangent, or the rectangle under its segments, if a secant, as the square of E H, if a tangent, or the rectangle under its segments, if a secant, to the square of F E, if a tangent, or the rectangle under its segments, if a secant. Consequently,

$$\left.\begin{matrix}\text{t. } AB^2\\ \text{or}\\ \text{s. } AB^r\end{matrix}\right\} : \left\{\begin{matrix}\text{t. } BG^2\\ \text{or}\\ \text{s. } BG^r\end{matrix}\right\} : \left\{\begin{matrix}\text{t. } CB^2\\ \text{or}\\ \text{s. } CB^r\end{matrix}\right.$$

$$\left.\begin{matrix}\text{t. } DE^2\\ \text{or}\\ \text{s. } DE^r\end{matrix}\right\} : \left\{\begin{matrix}\text{t. } EH^2\\ \text{or}\\ \text{s. } EH^r\end{matrix}\right\} : \left\{\begin{matrix}\text{t. } FE^2\\ \text{or}\\ \text{s. } FE^r\end{matrix}\right.$$

The square of A B therefore (22. v.) if a tangent, or the rectangle under its segments, if a secant, is to the square of C B, if a tangent, or the rectangle under its segments, if a secant, as the square of D E, if a tangent, or the rectangle under its segments, if a secant, to the square of F E, if a tangent, or the rectangle under its segments, if a secant.

BOOK I. If A B, C B, D E, F E be tangents, then it is evident (22. vi.) that A B : C B :: DE : F E.

PROP. X. Fig. 18, 19. 49. Any straight line (D C) parallel to a side (V B) of a cone (V A M B), provided it be not in the plane touching the cone in that side, will meet one of the opposite superficies in one point, and in one point only.

Let a plane pass through the parallels D C, V B, and as, by hypothesis, D C is not in the plane touching the cone in the side V B, the plane passing through D C, V B must cut the cone. Let it cut the cone in A V, B V and the opposite superficies in the straight lines A V, B V produced. Then as A V meets B V in V the vertex, and as it is in the same plane with the parallels V B, D C, (by ax. 12. i.) A V, or A V produced, must also meet D C. Let them meet in D. Then D C must meet one of the superficies in D, and as it is parallel to V B, it is evident it cannot meet the other superficies; for on one side of D it is entirely within one of the superficies, and on the other entirely without both.

PROP. XI. Fig. 18, 19. 50. If a straight line (D C) parallel to a side (V B) of the cone (V A M B) cut either of the conical superficies (in D) and meet two straight lines (E I F, L R T in the points E and R) parallel to the base of the cone, and which cut either superficies (in the points I, F and L, T); the segments (D E, D R) of the first mentioned line between the superficies and the points of concourse will be to one another as the rectangles (I E × E F, L R × R T) under the segments of the secants.

For through the parallels D C, V B let a plane pass, and let it cut the plane of the base in the straight line A C B, and the superficies in A V, B V. Through each of the straight lines L R T, E I F let a plane pass parallel to the base A M B, and let O L N T, I F H G be the circles formed, as in article 17. Let E G H, O R N be the intersections of these circles and the plane passing through A V, B V. Let E G H meet A V in G and B V in H; and let O R N meet A V in O and B V in N. Then (16. xi.) the straight lines E G H, O R N, A C B are parallel, and therefore E H, C B, R N are equal; and by similar triangles we have, BOOK I.

D E : D R : : E G : O R.

Hence (cor. 1. vi.) D E : D R : : E G × E H : O R × R N, and therefore (35. and 36. iii.)

D E : D R : : I E × E F : L R × R T.

51. If a straight line parallel to a side of the cone cut either of the opposite conical superficies, and meet two straight lines parallel to the base, and which meet either superficies; its segment between the superficies, and the first of the two parallel to the base, will be to its segment between the superficies and the second, as the square of the first, if a tangent, or the rectangle under its segments, if a secant, to the square of the second, if a tangent, or the rectangle under its segments, if a secant. For every thing remaining as above, if E P be parallel to the base, and touch either superficies in P, E P will be in the plane of the circle I F H G and (36. iii.) the square of E P will be equal to the rectangle under I E, E F. And, for the same reasons, if the point R were without the circle, the square of a straight line parallel to the base, drawn from R and touching either superficies, would be equal to the rectangle under L R, R T.

52. If a straight line (B C) cutting a parabola or hyperbola (in the point A) be parallel to a side of the cone in which the section is formed, and meet two straight lines (B D, C E) which are parallel to one another, and meet the same section or the opposite hyperbolas; its segment (A B) between the curve and PROP. XII. Fig. 13.

BOOK I. Fig. 13.

the first of the two parallels will be to its segment (A C) between the curve and the second, as the square of (B D) the first, if a tangent, or the rectangle under its segments, if a secant, to the square of (C E) the second, if a tangent, or the rectangle under its segments, if a secant.

If B D, C E be parallel to the base of the cone, this is evident from the last article; but if they are not, let B F, C G be parallel to the base, and let them touch or cut the same section or opposite hyperbolas. Then, by the last article,

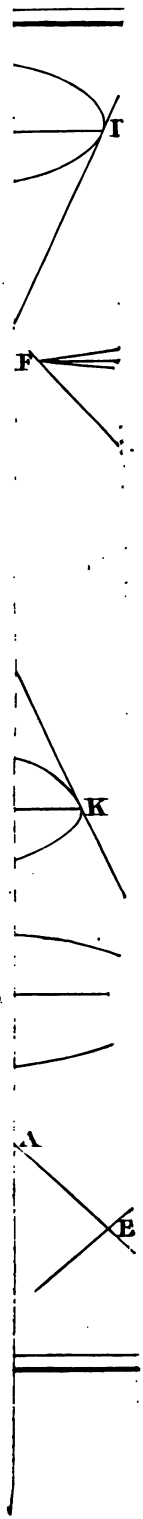
A B : A C :: t. BF^2 or s. BF^r : t. CG^2 or s. CG^r,

and by article 48, t. BF^2 or s. BF^r : t. CG^2 or

s. CG^r :: t. BD^2 or s. BD^r : t. CE^2 or s. CE^r.

Hence (11. v.)

A B : A C :: t. BD^2 or s. BD^r : t. CE^2 or s. CE^r.


I
F
K
A
E

ELEMENTS

OF

CONIC SECTIONS.

BOOK II.

Of the Ellipse and Hyperbola.

53. THAT point within an ellipse or between opposite hyperbolas, in which every straight line, passing through it and terminated by the curve or opposite curves, is bisected, is called *the Center* of the ellipse, or *the Center* of the hyperbola or opposite hyperbolas. BOOK II.

54. Any straight line passing through the center of an ellipse, and terminated by the curve, is called *a Diameter* of the ellipse.

55. A straight line passing through the center of opposite hyperbolas, and terminated by the opposite curves, is called *a Transverse Diameter* of the opposite hyperbolas, or of either of the opposite hyperbolas. And a straight line passing through the center of opposite hyperbolas, and bisecting a straight line not passing through the center, and terminated by the opposite curves, is called *a Second Diameter* of the opposite hyperbolas, or of either of the opposite hyperbolas.

56. Any straight line not passing through the center of an ellipse, or opposite hyperbolas, terminated by

BOOK II. the curve of the ellipse or either hyperbola, or by the opposite curves, and bisected by a diameter, is called *a Double Ordinate* to the bisecting diameter; and its half is simply called *an Ordinate* to it.

57. The points in which any diameter of an ellipse meets the curve, or in which any transverse diameter of opposite hyperbolas meets the opposite curves, are called *the Vertices* of the diameter; and the segments of a diameter, between an ordinate and its vertices, are called *Abscisses.*

58. Two diameters of an ellipse, or opposite hyperbolas, of which each bisects all straight lines terminated by the curve, or opposite curves, and parallel to the other, are called *Conjugate Diameters.*

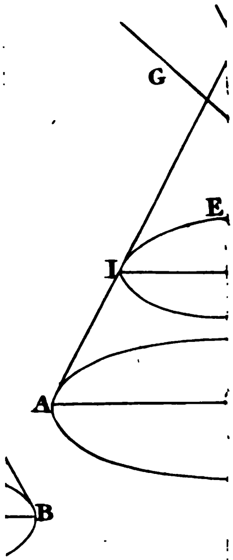
59. A diameter of an ellipse, or opposite hyperbolas, which cuts its ordinates at right angles is called *an Axis* of the ellipse, hyperbola, or opposite hyperbolas.

Fig. 20. 60. *Lemma.* If the points C, D be so situated in the straight line A B that the rectangle D A C is equal to the rectangle C B D, then A C is equal to B D: or if the rectangle A C B be equal to the rectangle B D A, then A C is equal to B D.

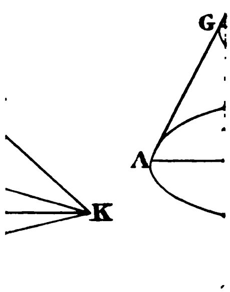
Case 1. Let C D be bisected in E, and then (6. ii.) $DAC + CE^2 = AE^2$, and $CBD + DE^2 = BE^2$, and therefore $AE^2 = BE^2$, $AE = BE$, and $AC = BD$.

Case 2. Let A B be bisected in E, and then (5. ii.) $ACB + EC^2 = AE^2 = BE^2 = BDA + ED^2$, and therefore $EC^2 = ED^2$, $EC = ED$, and $AC = BD$.

PROP. I. 61. If two parallel straight lines both touch or both cut, or one of them touch and the other cut, an ellipse, opposite hyperbolas, or an hyperbola, and if a straight line be drawn through the points of contact or bisecting the parallel secants, or through the point of contact and bisecting the secant parallel to the tangent, and if another straight line be drawn bisecting it, (in



Fig



C,) and parallel to the two first mentioned lines, each of the two lines thus drawn will bisect any secant parallel to the other. BOOK II.

Case 1. Let the parallel straight lines A B, F G touch an ellipse D H E K or opposite hyperbolas H I, K L in H and K, and, the straight line H K being drawn, let D E be drawn parallel to the tangents and bisecting H K in C; each of the two lines H K, D E thus drawn will bisect any secant parallel to the other. Fig. 21, 22.

Let I N be a secant parallel to D E, and if P be the point in which H K cuts it, it will be bisected in P.

For through I and N let B G, A F be drawn parallel to H K, and let the first of these meet the tangents in B, G the curve again in L, and D E in M, and the other meet the tangents in A, F the curve again in O, and D E in Q. Then, by article 48,

$$HB^2 : LB \times BI :: KG^2 : IG \times GL;$$

and as (34. i.) $HB = KG$, the rectangles (14. v.) L B I, I G L are equal; and therefore, by article 60, $GL = BI$. For the same reasons $FO = AN = BI$, and therefore $IL = NO$, and the rectangles L B I, O A N are equal. But, by article 48,

$$LB \times BI : HB^2 :: OA \times AN : HA^2,$$

and therefore (14. v.) $HB = HA$, and (34. i.) $PI = PN$.

If I L be any secant parallel to H K, and the rest be as already stated as to the parallelism of the lines in the figures, then as above $BI = GL$, and as $HC = KC$, $BM = GM$, and therefore I L is bisected in M.

The conclusion will be the same in the ellipse, if B G, A F touch the section. The only difference in the figure will be that L will coincide with I, and O with N, and in the demonstration that the rectangles L B I, L G I, O A N will become equal squares. Fig. 21.

Case 2. Let A B, F G be two parallel secants, and let the secant V O bisect them in H and K, and let Fig. 23, 24.

BOOK II. DE be parallel to AB, FG and bisect VO in C; each of the lines VO, DE will bisect any secant parallel to the other.

Let IN be a secant parallel to DE, and if P be the point in which HK cuts it, it will be bisected in P.

For through F, G let the straight lines FS, GT be drawn parallel to HKVO, and let the first of these meet AB in Q, IN in L, and the curve of the section or that of the opposite hyperbola in S. Let the other meet AB in R, IN in M, and the curve of the section or that of the opposite hyperbola in T. Then, by article 48,

$$AQ \times QB : FQ \times QS :: BR \times RA : GR \times RT,$$

and as $FK=GK$, (34. i.) $QH=RH$, and therefore the rectangles AQB, BRA are equal, and (14. v.) $FQS=GRT$. Hence (cor. 1. vi.) $QS=RT$, and $FS=GT$, and the rectangles FLS, GMT are equal. But, by article 48,

$$FL \times LS : IL \times LN :: GM \times MT : NM \times MI,$$

and therefore (14. v.) $IL \times LN = NM \times MI$, and by article 60, $IL=NM$; and as FK, GK are equal, we have $LP=MP$, and IN is bisected in P.

Let FS be now considered as any secant parallel to VO, and meeting DE in X, and let the secants FG, ST, drawn through F, S and parallel to DE meet VO in K, W. Then, as above, FG is bisected in K and ST in W. By article 48,

$$VW \times WO : SW^2 :: OK \times KV : FK^2,$$

and as $SW=FK$, $VW \times WO = OK \times KV$, and by article 60, $OW=VK$, and therefore $CW=CK$, and consequently (34. i.) FS is bisected in X.

Fig. 25, 26. Case 3. Let the straight line AB touch the ellipse or hyperbola in H, and FG being a secant parallel to it, let HL bisect FG in K, and let the secant HL be bisected in C by DE parallel to AB or FG; then each of the lines HL, DE will bisect any secant parallel to the other.

For let NO be a secant parallel to AB, FG, and meet HL in P, and through F and G let AI and BM be drawn parallel to HL, and let the first of these meet the tangent in A, DE in S, NO in Q, and the curve of the section or opposite hyperbola in I. Let the other meet the tangent in B, NO in R and the curve of the section or opposite hyperbola in M. Then, by article 48,

$$HA^2 : IA \times AF :: HB^2 : MB \times BG,$$

and as FK=KG, (34. i.) HA=HB, and therefore (14. v.) the rectangles IAF, MBG, are equal, and (cor. 1. vi.) FI=GM. The rectangles FQI, GRM are therefore equal; and as by article 48,

$$FQ \times QI : NQ \times QO :: GR \times RM : OR \times RN,$$

the rectangles NQO, ORN (14. v.) are equal, and by article 60, NQ=OR, and NO is bisected in P.

Let FI be now considered as any secant parallel to HL, and meeting DE in S, and let the secants FG, IM, drawn through F, I and parallel to DE, meet HL in K, T. Then, as above, FG is bisected in K and IM in T. By article 48,

$$FK^2 : HK \times KL :: IT^2 : LT \times TH,$$

and therefore (14. v.) the rectangles HKL, LTH are equal, and, by article 60, HK=LT, and KC=CT, and therefore (34. i.) FI is bisected in S.

62. If two parallel straight lines both cut, or one of them touch and the other cut, an ellipse, hyperbola, or opposite hyperbolas, a straight line bisecting the parallel secants, or passing through the point of contact and bisecting the secant, will cut the curve of the section or opposite hyperbola in that point in which a straight line parallel to the two first mentioned touches the section or opposite hyperbola.

BOOK II. For, by the second case of the last article, V O bisects any secant parallel to A B, F G, and therefore

Fig. 23, 24. straight lines drawn through V and O in Figs. 23, 24, parallel to A B, F G must touch the ellipse or opposite hyperbolas. By the third case H L in Figs. 25, 26, bisects any secant parallel to A B, F G; and therefore a straight line parallel to them and passing through L must touch the ellipse or opposite hyperbola. Lastly, in the ellipse, and in the first case of the last article, as D E, in Fig. 21, bisects any secant parallel to H K, it is evident that straight lines parallel to H K and passing through D, E must touch the ellipse.

PROP. II. 63. Every thing being as stated in article 61, the point of bisection C is the center of the ellipse or opposite hyperbola; and no other point can be the center.

Fig. 21, 22. For from any point L in the curve, let the secant L I be drawn parallel to H K, and let it meet D E in M; and from I let the secant I N be drawn parallel to D E, and let it meet H K in P. Then, by article 61, L I is bisected in M, and I N in P, and therefore, N P : P C : : N I : I L. Consequently, as P C, I L are parallel, if L C be drawn and produced, it will pass through N, and by similar triangles, N P : N C : : N I : N L, and N L is bisected in C; and therefore, according to article 53, C is the center.

Nor can any other point be the center. This is evident in the ellipse; for a secant passing through C cannot be bisected in any other point. In the hyper-

Fig. 24. bola let the point X in D E be supposed to be a center, and let every thing be in the figure as stated for Case 2, article 61, and then F S is bisected in X. Let the secant Y I pass through X, and let it meet F G in Z and T S in *b*. Then as F Z, S *b* are parallel, and as X F, X S are equal, X Z is equal (29. 26. i.) to X *b*. Con-

sequently X Y is less than X b, and therefore much less than X I, and the secant Y I is not bisected in X. The point X therefore cannot be the center; and it is evident that no point out of D E can be the center. BOOK II.

64. If in an ellipse or hyperbola one secant (A B) be a double ordinate to a diameter, (V O,) any other secant (I N) parallel to it will be a double ordinate to the same diameter, and tangents passing through the vertices of the diameter are parallel to the ordinates. Fig. 23, 24.

For, by article 61, if V O bisect A B in H, and I N in P, it will bisect any other secant parallel to them; and, by article 63, it will pass through C the center. Consequently, as the diameter V O passes through H, the point in which A B is bisected, it must pass through P, the point in which I N is bisected. It is also evident, by article 62, that tangents passing through V, O must be parallel to A B, an ordinate to V O.

65. In an ellipse and hyperbola, or opposite hyperbolas, two secants, not passing through the center, cannot bisect one another.

For if two secants, not passing through the center, could bisect one another, by the last article, a straight line drawn parallel to each through the vertex of that diameter which passes through the point of bisection, would touch the section; and this, by article 32, is impossible.

66. Two diameters of an ellipse or opposite hyperbolas, are conjugate diameters, if one of them be parallel to the ordinates of the other, or to tangents passing through its vertices; and ordinates to a diameter, tangents passing through its vertices, and its conjugate diameter are parallel to one another. PROP. III.

The first part of this is evident from articles 61, 62, 63, and 58; and the second part is evident from the first part and articles 64, 65.

67. If C be the center of the opposite hyperbolas Fig. 27.

BOOK II. PA, FB, and AB a transverse diameter, to which DE is the conjugate, and HF an ordinate, and if the rectangle under AH, HB be to the square of HF as the square of CB to the square of CE or CD, the points D, E are called *the Vertices of the second Diameter* DE. In this way the magnitude of any second diameter is determined by its vertices.

PROP. IV. 68. If each of two straight lines, meeting one another, touch or cut, or one of them touch, and the other cut, an ellipse, hyperbola, or opposite hyperbolas; the square of the first of the two, if a tangent, or the rectangle under its segments, if a secant, will be to the square of the second, if a tangent, or the rectangle under its segments, if a secant, as the square of the semi-diameter parallel to the first to the square of the semi-diameter parallel to the second.

In the ellipse, and when the two straight lines are parallel to two transverse diameters of opposite hyperbolas, the Proposition is evident from article 48. For diameters in the ellipse, and transverse diameters of opposite hyperbolas, are secants meeting one another in the center, in which they are bisected.

In the hyperbola the property may be extended to second diameters in the following manner.

Fig. 27. Let PA, FB be two opposite hyperbolas, of which C is the center, and let the two secants GF, MO cut one another in I. Let GF be bisected in H and MO in N, and through these points let the diameters ABH, RSN be drawn, and let the secants ILQ, IKP be drawn parallel to them. Then CE being half the conjugate diameter to AB, and CT half that to RS, by articles 48 and 67, we have the following proportions,

$$GI \times IF : QI \times IL :: HF^2 : AH \times HB :: CE^2 : CB^2,$$

$$QI \times IL : PI \times IK :: CB^2 : CS^2,$$

$$PI \times IK : MI \times IO :: RN \times NS : NO^2 :: CS^2 : CT^2.$$

Hence, placing for ex æquali, BOOK II.

$GI \times IF : QI \times IL : PI \times IK : MI \times IO$

$CE^2 : CB^2 : CS^2 : CT^2$,

and therefore (22. v.)

$GI \times IF : MI \times IO :: CE^2 : CT^2$.

From these proportions and articles 48, 67, the truth of the assertions in this, in every case, is evident.

69. If two straight lines be ordinates to any diameter of an ellipse, or transverse diameter of an hyperbola, by the last, and article 66, (and 11. v.) the square of the first will be to the square of the second, as the rectangle under the abscisses corresponding to the first to the rectangle under the abscisses corresponding to the second.

70. From article 68 (and 22. vi.) it is evident, that if two straight lines meeting one another touch an ellipse, hyperbola, or opposite hyperbolas, they will be to one another as the semidiameters to which they are parallel.

71. From article 68, it is evident, that if two conjugate diameters of an ellipse cut one another at right angles, they cannot be equal to one another; for if they were equal to one another, the section would be a circle, as the square of the ordinate would be equal to the rectangle under the corresponding abscisses.

72. A straight line which is a third proportional to two conjugate diameters of an ellipse, or opposite hyperbolas, is called the *Parameter*, or *Latus Rectum*, of that diameter which is the first of the three proportionals.

73. If a straight line (G F) be an ordinate to any diameter (A B) of an ellipse, or any transverse diameter of an hyperbola, the rectangle (A F B) under the abscisses of the diameter will be to the square of the ordinate (GF^2) as the diameter to its parameter. PROP. V. Fig. 28, 29.

Let C be the center of the ellipse or hyperbola, and let D E be the diameter parallel to G F, and consequently, by article 66, the conjugate diameter to A B, and let B H be the parameter of A B. Then we have the following proportions,

By article 72, $AB : DE :: DE : BH$.
(Cor. 2. 20. vi.) $AB^2 : DE^2 :: AB : BH$.
(15. v.) $AB^2 : DE^2 :: CB^2 : CD^2$, and
therefore (11. v.) $CB^2 : CD^2 :: AB : BH$, and
by article 68, $CB^2 : CD^2 :: AF \times FB : GF^2$,
consequently (11. v.) $AF \times FB : GF^2 :: AB : BH$.

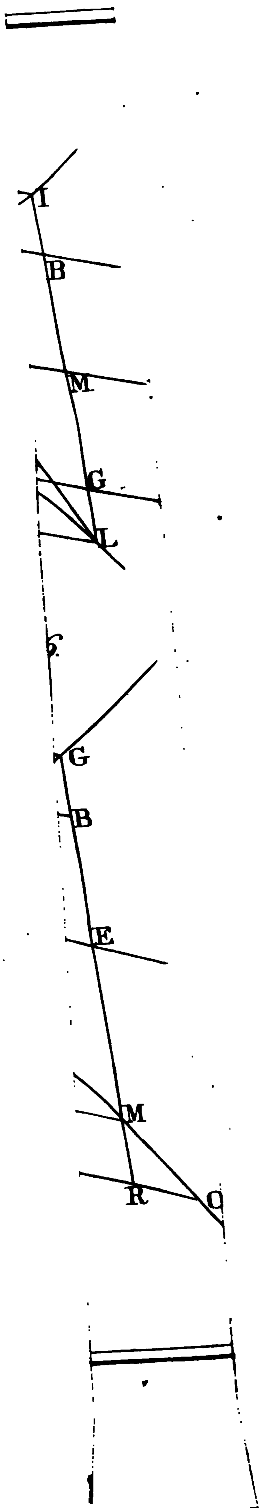
74. The rest remaining as in the last article, let the parameter B H be at right angles to the diameter A B, and from the other vertex A, draw A H. From the point F draw F K perpendicular to A B, and let it meet A H, or A H produced, in K. Complete the rectangle K B, and it will be equal to the square of the ordinate F G. For, as B H, F K are at right angles to A B, they are parallel; and therefore, by the last article, (and 4. vi.)

$$AF : FK :: AB : BH :: AF \times FB : GF^2.$$

But (1. vi.) $AF : FK :: AF \times FB : FK \times FB$.

Hence (11. and 14. v.) $FK \times FB = GF^2$.

75. Complete the rectangle L A B H, and let L H meet F K in M, and let K N, the side of the rectangle K B, opposite to B F, meet B H in N; then in the ellipse the square of the ordinate F G is less than the rectangle under the absciss F B and the parameter B H, by the rectangle M N, similar to L B, and having one of its sides equal to B F; but in the hyperbola, the square of the ordinate F G, is greater than the rectangle under the absciss B F and the parameter B H, by the rectangle M N similar to L B, and having one of its sides equal to B F. This is evident from the last article.


I
B
M
G
L
6.
G
B
E
M
R
O

76. *Scholium*. On account of the deficiency of the square of F G from the rectangle under F B, B H in Fig. 28, Apollonius * called the section an ellipse; and on account of the excess of the square of F G above the rectangle under F B, B H in Fig. 29, he called the section an hyperbola. BOOK II. Fig. 28, 29.

From the properties demonstrated in the last two articles, these sections are frequently denoted by Algebraical equations, in the following manner. Put the diameter A B (in Fig. 28. and 29.) $=a$, its parameter B H $=p$, the absciss F B $=x$, and the ordinate F G $=y$. Then A F $=a\mp x$, the negative sign applying to the ellipse, and the positive sign to the hyperbola. And by the similar triangles A B H, A F K, $a : p :: a\mp x : \frac{ap\mp px}{a}=$ F K. Consequently, by article 74,

$$\frac{ap\mp px}{a}\times x=px\mp\frac{px^2}{a}=px\mp\frac{p}{a}x^2=y^2.$$

77. If a straight line (B D) touching the section (in E) meet any diameter (A I) of an ellipse (A E I) or transverse diameter (A I) of an hyperbola (E I G), and from the point of contact an ordinate (E F) be drawn to the diameter (A I), the abscisses (A F, F I) will be to one another as the distances (A M, I M) of the point of concourse (M) from the vertices of the diameter. PROP. VI. Fig. 30, 31.

Let C be the center, and A B, I D the parallel tangents passing through the vertices A, I, and let them meet B D in B and D.

We have then the following proportions:

Article 48, E B : E D :: A B : I D,

* Apollonius was born at Perga, in Pamphylia, about 240 years before Christ. His Treatise on Conic Sections, consisting of eight books, is the most early work on the subject which has come down to the present time; and the most complete edition of it was published at Oxford, in the year 1710, by Dr. Halley.

BOOK II. but (10. vi.) $EB : ED :: AF : FI$,
and (4. vi.) $AB : ID :: AM : IM$.
Hence (11. v.) $AM : IM :: AF : FI$.

PROP. VII. 78. If a straight line (EM) touching an ellipse or
Fig. 30, 31, 32. hyperbola (in the point E) meet a diameter (in M), and from the point of contact there be drawn an ordinate (EF) to the diameter (AI), the semidiameter (CI) will be a mean proportional between the segments (CF, CM) of the diameter between the center and ordinate, and between the center and tangent.

Fig. 30, 31. First let every thing be as in the last article, and then we have the following proportions:

(18. v.) $AM+IM : IM :: AF+FI : FI$.

By halving the antecedents,

in the ellipse $CM : IM :: CI : FI$,
in the hyperbola $CI : IM :: CF : FI$.

Consequently, by conversion,

in the ellipse $CM : CI :: CI : CF$,
in the hyperbola $CI : CM :: CF : CI$, and by
inversion, $CM : CI :: CI : CF$.

Fig. 32. Secondly, let AI be a second diameter of the opposite hyperbolas GK, EL, and EF an ordinate to it; and let the tangent EM meet it in M, and the transverse diameter KL parallel to EF in N, and let EP be an ordinate to KL. Then,

by the above, $CP : CL :: CL : CN$,
(Cor. 2. 20. vi.) $CP^2 : CL^2 :: CP : CN$,
(4. vi.) EF or $CP : CN :: MF : CM$,
(11. v.) $CP^2 : CL^2 :: MF : CM$,
(17. v. and 6. ii.)
$KP \times PL : CL^2 :: CF : CM :: CF^2 : CF \times CM$,
(16. v.) $KP \times PL : CF^2$ or $EP^2 :: CL^2 : CF \times CM$.
By article 67, $KP \times PL : EP^2 :: CL^2 : CI^2$, and
therefore (11. and 9. v.) $CF \times CM = CI^2$.
Consequently $CM : CI :: CI : CF$.

79. From the above, (and 17. vi.) $CM \times CF = CI^2$, and therefore in the ellipse $CM^2 - CI^2 = CM^2 - CM \times CF =$ (6. ii.) $AM \times MI =$ (2. ii.) $CM \times MF$. BOOK II. Fig. 30, 31.

But when AI is a transverse diameter in the hyperbola, $CM \times CF - CM^2 = CI^2 - CM^2 =$ (3. ii.) $CM \times MF =$ (5. ii.) $AM \times MI$.

Again, in the ellipse, $CM \times CF - CF^2 = CI^2 - CF^2 =$ (3. ii.) $CF \times FM =$ (5. ii.) $AF \times FI$. But AI being a transverse diameter of the hyperbola, $CF^2 - CM \times CF = CF^2 - CI^2 =$ (2. ii.) $CF \times FM =$ (6. ii.) $AF \times FI$.

80. If two straight lines (EM, GM) touching an ellipse, hyperbola, or opposite hyperbolas, meet one another (in M), the diameter (AI) bisecting the line (EG) joining the points of contact will pass through (M) the point of concourse. PROP. VIII. Fig. 30, 31, 32.

For let C be the center, and let A, I be the vertices of the diameter; and then, as EG is bisected by the diameter AI, EF is an ordinate to it, and therefore, by article 78, CF : CI :: CI : the segment of the diameter intercepted between C and the tangent EM. For the same reasons CF : CI :: CI : the segment of the diameter intercepted between C and the tangent GM. Consequently the segment of the diameter between C and the tangent EM, is equal to the segment of the diameter between C and the tangent GM. The diameter must therefore pass through M; for if it did not, it would, upon being produced, first meet the one tangent, and then the other, and its segments between C and the tangents would be unequal.

81. From the last it is evident that if two straight lines touching an ellipse, hyperbola, or opposite hyperbolas, meet one another, a straight line passing through the point of concourse, and bisecting the line joining the points of contact, will be a diameter.

BOOK II. PROP. IX. Fig. 33, 34, 35. 82. If two parallel straight lines (A B, I D) touching an ellipse or opposite hyperbolas, meet a third tangent (B D), the rectangle (A B × I D) under their segments, between the points of contact and the points of concourse, will be equal to the square of the semidiameter to which they are parallel, and the rectangle (B E D) under the segments of the third tangent, between its point of contact (E) and the parallel tangents, will be equal to the square of the semidiameter to which it is parallel.

For let C be the center, and draw E G parallel to A B, I D; and draw also A I. Then by articles 63, 61, A I is a diameter, and, by article 66, E G is either an ordinate to A I, or in the ellipse the conjugate diameter to it.

Fig. 33. First, let E G be the conjugate diameter to A I, and then, by article 66, B D, A I are parallel to one another, and also A B, C E, I D to one another. Consequently (34. i.) A B, C E, I D are equal to one another, as are also A C, C I, B E, E D to one another; and therefore $AB \times ID$ is equal to CE^2, and $BE \times ED$ is equal to AC^2.

Fig. 34, 35. Next, let E G be an ordinate to the diameter A I, and let it meet it in F. Let C K be the semidiameter parallel to the tangents A B, I D, and C P the semidiameter parallel to the tangent E B; and let E B meet the diameter A I in M, and the diameter K C L in H. Let E O be an ordinate to K C L, and let it meet it in O. Then, by article 66, A B, L H, G E, I D are parallel, and E O is parallel to A I. Then, we have, by article 70,

$$BE : AB :: CP : CK,$$

$$\text{and } ED : ID :: CP : CK, \text{ and,}$$

article 44, $$BE \times ED : AB \times ID :: CP^2 : CK^2.$$

But, by article 78, (and 16. vi.)

$$CK^2 = CH \times CO = CH \times EF,$$ BOOK II.

and by article 79, and (16. and 4. vi.)

$AM : CM :: MF : MI :: AB : CH :: EF$ or $CO : ID$.

Consequently, $AB \times ID = CH \times CO = CK^2$,

and (14. v.) $BE \times ED = CP^2$.

83. Every thing being as in the last article,

$$ME \times EH = BE \times ED = CP^2.$$

For, by article 79, (and 16. and 10. vi.)

$$AF : FM :: CF : FI :: BE : EM :: EH : ED,$$

and therefore (16. vi.) $ME \times EH = BE \times ED = CP^2$.

84. If a straight line as BE, touching an ellipse or hyperbola in the point E, meet the diameter AI in M, and the diameter KL in H, and if the rectangle under ME, EH be equal to the square of the semidiameter parallel to BE, the diameters AI, KL will be conjugate to one another.

85. To find the axes of a given ellipse or hyperbola, (DBE) the center (C) being also given; and to demonstrate that the same section can have only two axes. PROP. X. Fig. 36, 37.

Part I. In the ellipse draw CG, CF two semidiameters, and, if they be unequal, let CG be greater than CF. With C as a center, and a distance less than CG but greater than CF, describe the circle HDE. Then from this construction and the nature of the two curves it is evident, that the circumference of the circle will cut the curve of the ellipse in four points, two of them being towards the left of the center, as the figure is viewed, and two of them towards the right. Let the circumference of the circle cut the curve of the ellipse in the points D, E. Draw the straight line DE; and through C draw AB bisecting DE in I, and (3. iii.) AB will be at right angles to DE. Through C draw LM parallel to DE, and AB, Fig. 36.

BOOK II. L M will be the axes of the ellipse, as is evident by articles 66 and (3. iii.) 59.

If the semidiameters C G, C F be equal, then a diameter bisecting the angle G C F will be one of the axes, and a diameter at right angles to it will be the other.

Fig. 37. Next, let the section D B E be an hyperbola, of which C is the center, and let K be any point within the hyperbola. With C as a center, and C K as a distance, describe the circle E K D, and let its circumference cut the curve of the hyperbola in the points E, D. Draw D E and bisect it in I, and through I draw the diameter A B; and parallel to D E draw the diameter L M. The diameters A B, L M are the axes of the hyperbola. For D E is a double ordinate to the diameter A B, and (3. iii.) A B cuts it at right angles, and L M is parallel to the ordinate D E.

Fig. 38. Part II. To demonstrate that an ellipse or hyperbola can have only two axes. First, let the section B D A be an ellipse, and C being the center, let R L, F G be the axes, found as above; and, if it be possible, let the diameter A B be also an axis. Let L D be a double ordinate to A B, meeting it in E, and the curve again in D; and the diameter D K being drawn, let D H be an ordinate to R L. Then, as by hypothesis A B is an axis, C E L, C E D are right angles, and as L D is bisected in E, (4. i.) C L is equal to C D. Again, as by the above R L, F G are conjugate, D H is parallel to F G, by article 66, and by article 68, $CL^2 : CF^2 :: RH \times HL : DH^2$. But, by article 71, C L, C F must be unequal, and therefore, supposing C L to be the greatest, CL^2 is greater than CF^2, and $RH \times HL$ greater than DH^2. To these unequals add CH^2, and then (5. ii. and 47. i.) CL^2 is greater than

Fig.

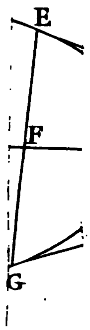
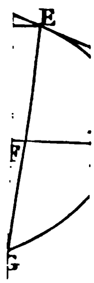
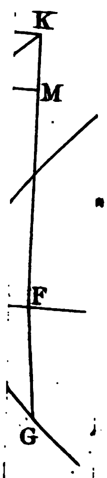


Fig. 3.





CD^2; and consequently CL is greater than CD. But CD is also equal to CL: which is absurd. The diameter AB therefore cannot be an axis. BOOK II.

Next, let the section EBD be an hyperbola, of which C is the center, FA the opposite hyperbola, and AB, LM the axes found as above; and, if it be possible, let the transverse diameter DF be an axis. Let DT touch the hyperbola in D, and meet the axis AB in T; and let DIE be an ordinate to AB, and let it meet it in I. Then in the triangle CID, CID is a right angle, by article 59; and therefore CDI is less than a right angle, and consequently CDT is much less than a right angle. But, by article 66, the tangent TD is parallel to the ordinates of the axis FD; and therefore, by article 59, (and 29. i.) the angle TDC is a right one. And, by the above, it is also less than a right one: which is absurd. Consequently DF is not an axis. Nor can a second diameter as GH, besides LM, be an axis. For FD conjugate to GH being drawn, and DT a tangent, the demonstration would end in the same absurdity. Fig. 39.

It is evident, that the axes of an ellipse, or hyperbola are conjugate diameters.

86. Of all the diameters of an ellipse the greater axis is the greatest and the lesser axis is the least; and of opposite hyperbolas the axes are the least diameters of the same kind. PROP. XI.

Part I. Let ABD be an ellipse, of which C is the center, RL the greater axis, and FG the lesser axis; of all the diameters RL is the greatest and FG the least. Fig. 38.

For let KD be any other diameter, and let DH be an ordinate to RL, and DM an ordinate to FG. Then, by article 66, DH is parallel to FG, and DM to RL; and therefore, by article 68, $CL^2 : CF^2 :: RH \times$

BOOK II. $HL : DH^2$, and as CL is greater than CF, CL^2 is greater than CF^2, and $RH \times HL$ is greater than DH^2. To these add CH^2, and (5. ii. and 47. i.) then CL^2 is greater than CD^2. Consequently CL is greater than CD, and therefore RL is greater than KD. Again, by article 68, $CL^2 : CF^2 :: DM^2 : GM \times MF$, and therefore, as above, DM^2 is greater than $GM \times MF$. To these add the square of CM, and then (5. ii. and 47. i.) CD^2 is greater than CF^2. Consequently KD is greater than FG.

Fig. 39. Part II. Let AB, LM be the axes of the opposite hyperbolas EBD, AF, and FD, GH any other conjugate diameters; then AB is less than the transverse diameter FD, and LM is less than GH.

For let DE be an ordinate to the axis AB, and let it meet it in I, and let DT touch the hyperbola in D, and let it meet AB in T. Let BP touch the hyperbola in the vertex B, and let it meet the tangent DT in P, and let C be the center. Then,

by article 78, $CI : CB :: CB : CT$,
by conversion, $CI : BI :: CB : BT$,
and therefore (14. v.) BI is greater than BT, and (2. vi. and 14. v.) DP greater than PT. Hence as PBT is a right angle, and therefore (19. i.) PT greater than BP, DP is greater than BP. Consequently it is evident that GH is greater than LM; for, by article 70,

$$DP : BP :: CH : CM.$$

87. In the ellipse, the greater axis is called the *Transverse Axis*, and frequently the *Major Axis*, or *Focal Axis*; and the other is called the *Conjugate Axis*, or *Minor Axis:* and in the hyperbola, the axis, which is a transverse diameter, is called the *Transverse*, or *Focal Axis*; and the other is called the *Conjugate Axis*.

Fig. 40, 41. 88. If C be the center, AB the transverse axis, and DE the conjugate axis of the ellipse ADB, or of the opposite hyperbolas A I, B P, then if in A B two points F, O be so taken that the rectangle under A F, F B, and also the rectangle under A O, O B, be equal to the square of C D or C E, the semiconjugate axis; the points F, O are called the *Foci*, or *Umbilici*, of the ellipse, hyperbola, or opposite hyperbolas.

Hereafter, unless it is otherwise specified, C denotes the center of an ellipse or hyperbola, A B the transverse, and D E the conjugate axis, and F, O the foci.

89. As (axiom i. 1.) A F × F B is equal to A O × O B, the foci F, O are equally distant from the vertices A, B by article 60. It is also evident, that the foci are equally distant from the center.

90. In the ellipse the distance of each of the foci from either extremity of the conjugate axis is equal to the semitransverse axis. For, supposing a straight line to be drawn from D to O, the square of D O (47. i.) will be equal to the squares of C O, C D together; and therefore, by article 88, (and 5. ii.) the square of D O is equal to the square of A C. Consequently D O is equal to A C; and therefore if with D or E as a center, and A C or C B as a distance, a circle be described, the circumference will cut A B in O and F, the foci. Fig. 40.

91. In the hyperbola the distance of each of the foci from the center is equal to the distance between the vertices of the transverse and conjugate axes. For, supposing a straight line to be drawn from D to A, the square of D A (47. i.) will be equal to the squares of C A, C D together; and therefore, by article 88, (and 6. ii.) the square of D A is equal to the square of C O or C F. Consequently C O or C F is equal to D A; and therefore the foci F, O may be easily found from the axes. Fig. 41.

BOOK II. 92. The double ordinate T S to the axis A B, drawn through either focus, suppose F, is equal to the para-
Fig. 40, 41. meter of the axis A B. For,

by article 68, $CB^2 : CD^2 :: AF \times FB$ or $CD^2 : TF^2$;
and (22. vi.) CB : CD :: CD : TF.
Hence (15. vi.) AB : DE :: DE : TS, and therefore, by article 72, T S is equal to the parameter of A B.

93. If through the foci F, O of an ellipse A D B, or of the opposite hyperbolas A I, B P, ordinates F T, O I to the axis A B be drawn, and if through the points T, I in which they meet the curve straight lines H T, I L be drawn to touch the section, or opposite hyperbolas, the tangents H T, I L are called *Focal Tangents*.

PROP. XII. Fig. 40, 41. 94. If a tangent (A H) passing through a vertex (A) of the transverse axis of an ellipse or hyperbola (A I), meet a focal tangent (H G), its segment (A H) between the point of contact (A) and point of concourse (H) will be equal to (A F) the segment of the axis between the point of contact and the focus (F) to which the focal tangent belongs.

For let H G touch the section in T, and T F, being drawn, will be an ordinate to A B, by art. 93. Let B G touch the section in B and meet H G in G, and then, by art. 66, A H, D E, T F, B G are parallel. Consequently,

art. 48, AH : BG :: HT : TG,
(10. vi.) HT : TG :: AF : FB,
(11. v.) AH : BG :: AF : FB.

As, therefore, the rectangles $AH \times BG$, $AF \times FB$ are similar, and, by articles 82, 88, equal to one another, A H is equal to A F, and B G to B F.

95. If O I be drawn an ordinate to A B, and on the side of A B opposite to that on which F T is, and if through I, the point in which it meets the curve, there

be drawn the focal tangent K L, meeting the tangent HA in K, and the tangent G B in L; then H L will be a parallelogram, and each of the opposite sides H K, G L will be equal to the transverse axis A B. For, by the above, and art. 89, A K, A O, B F, B G are equal to one another, and also A H, A F, B L, B O to one another. Consequently H K, G L are equal and parallel, and therefore (33. i.) H G, K L are equal and parallel. Hence H L is a parallelogram; and as A H is equal to A F, and A K to B F, H K or G L is equal to A B.

PROP. XIII. Fig. 40, 41.

96. If from any point (P) in the curve of an ellipse or hyperbola (P T B) two straight lines (P F, P O) be drawn to the foci, their sum in the ellipse, but their difference in the hyperbola, will be equal to the transverse axis.

For, the rest remaining as in the two last articles, let P R be drawn an ordinate to A B, and let it meet the curve again in M, the focal tangent G H in N, and the focal tangent K L in Q, and then,

by art. 48, $TH^2 : TN^2 :: AH^2 : MN \times NP$,

(10. vi.) $TH^2 : TN^2 :: AF^2 : FR^2$,

(11. v.) $AH^2 : MN \times NP :: AF^2 : FR^2$.

Consequently, by art. 94, (and 14. v.) $MN \times NP = FR^2$, and $MN \times NP + PR^2 = FR^2 + PR^2 =$ (6. ii. and 47. i.) $RN^2 = FP^2$, and $RN = FP$. For the same reasons $OR^2 = PQ \times QM$. To these equals add the square of R M, or its equal the square of R P, and then (6. ii. and 47. i.) the square of O M in the ellipse, and the square of O P in the hyperbola is equal to the square of R Q. But in the ellipse (4. i.) $OM = OP$, and therefore in each section $OP = RQ$. Consequently,

in the ellipse, $RN + RQ = NQ = HK = FP + OP$,

in the hyperbola, $RQ - RN = NQ = HK = OP - FP$;

BOOK II.

and therefore, by article 95,

in the ellipse, $FP+OP=AB$,

in the hyperbola, $OP-FP=AB$.

97. If from any point in the curve of an ellipse, or hyperbola, two straight lines be drawn to the foci, in the ellipse the difference between the transverse axis and either of the two will be equal to the other; but in the hyperbola the sum of the transverse axis and the least of the two will be equal to the other.

98. If the conjugate axis D E be produced till it meet the opposite focal tangents in V and W, each of the segments C V, C W will be equal to the semi-transverse axis. For, let the opposite focal tangents meet the transverse axis A B in X and Y. Then, as, by art. 89, C F, C O are equal, it is evident, from art. 78, that C Y, C X are equal; and as X W, V Y are parallel, the angles (29 i.) C X W, C Y V are equal. Consequently, as the angles at C are right angles, (26 i.) C V is equal to C W.

PROP. XIV. Fig. 42, 43.

99. If from a point (H) without an ellipse or opposite hyperbolas two straight lines (H F, H O) be drawn to the foci, their sum in the ellipse will be greater, but their difference in the hyperbola will be less, than the transverse axis. But if from a point (G) within an ellipse or hyperbola two straight lines (G F, G O) be drawn to the foci, their sum in the ellipse will be less, but their difference in the hyperbola greater, than the transverse axis.

Part I. Let H O cut the curve of the ellipse or hyperbola in P, and let F P be drawn; and then in the ellipse,

(20. i.) $FH+HP>FP$, and therefore,

$FH+HP+PO$ or $FH+HO>FP+PO$ or AB, by article 96.

In the hyperbola let F H be greater than H O, and then

(20. i.) $FH < FP + PH$ that is,
by art. 97, $FH < AB + OP + PH$, or $FH < AB + HO$,
and therefore $FH - HO < AB$.

Part II. Let F G meet the curve of the ellipse or hyperbola in P, and let O P be drawn; and then in the ellipse,

(20. i.) $GO < OP + PG$,
and therefore $GO + GF < OP + PG + GF$.
Hence $GO + GF < OP + PF$ or A B, by art. 96.

In the hyperbola,
(20. i.) $OP + PG > GO$, that is,
by article 97, $FP - AB + PG$ or $FG - AB > GO$.
Hence $FG > GO + AB$, and therefore
$FG - GO > AB$.

100. From the last and article 96, it is evident, that two straight lines being drawn from any point to the foci of an ellipse or hyperbola, if in the ellipse their sum be greater, or in the hyperbola their difference be less, than the transverse axis, the point will be without the section. If in the ellipse their sum or in the hyperbola their difference be equal to the transverse axis, the point will be in the curve of the section. And lastly, if in the ellipse their sum be less, or in the hyperbola their difference be greater, than the transverse axis, the point will be within the section.

101. If O, F be the foci of the hyperbola B P, and if the side O G, of the triangle O G F, be equal to A B, and O G F be an obtuse angle, then the straight line O G produced will meet the curve of the hyperbola B P, in which the focus F is situated. For let O G be produced to K, and make the angle G F L equal to the angle F G K. Then, as by hypothesis O G F is an obtuse angle, K G F is an acute angle, and therefore, as the angle G F L is equal to it, the straight lines G K, F L, being produced, will meet. Let them meet in Fig. 44.

BOOK II. P, and (6. i.) P F will be equal to P G. Consequently, as the difference of P O, P F is equal to O G, or A B, the point P is in the curve of the hyperbola, by the last article.

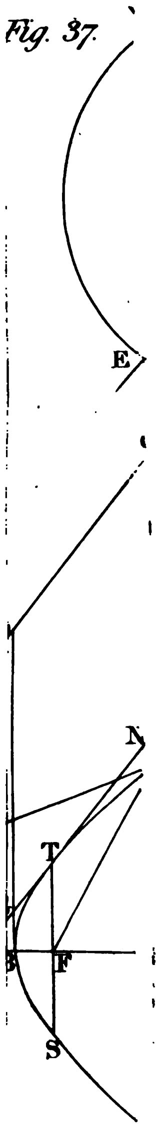
PROP. XV. Fig. 45, 46. 102. If from any point (P) in the curve of an ellipse, or hyperbola, two straight lines be drawn to the foci, the straight line (H P) bisecting the angle (K P F) adjacent to that contained by them will touch the ellipse; but the straight line (H P) bisecting the angle (F P O) contained by them will touch the hyperbola.

For the ellipse let P K be taken in O P produced, equal to P F, but for the hyperbola let P K be taken in P O, equal to P F, and then, by article 96, O K=A B. Let F K be drawn, and let P H, bisecting the angle F P K, meet it in H. Let G be any point in P H, or in P H produced, and let G K, G F, G O be drawn. Then (4. i.) F H=H K, and the angles F H P, K H P are equal, and therefore (4. i.) F G=K G. Consequently as in the ellipse (20. i.) K G+G O>O K, we have G F+G O>A B; and as in the hyperbola (20. i.) G K+O K>G O, we have G O−G K or G O−G F< A B. In either case therefore the point G, by article 100, is without the curve, and consequently P H touches it in P.

103. From the last and article 32, it is evident, that if a straight line touch an ellipse or hyperbola, and straight lines be drawn from the point of contact to the foci, in the ellipse the tangent will bisect the angle adjacent to that contained by these two straight lines drawn to the foci; but in the hyperbola the tangent will bisect the angle contained by these two straight lines drawn to the foci. In the ellipse the angle O P G (15. i.) is equal to the angle F P H. Fig. 45.

Fig. 45, 46. 104. If from the foci O, F of an ellipse, or hyperbola, two straight lines, O K, F K be drawn to a third

Fig. 37.



point K, of which O K, one of them, is equal to the transverse axis A B, and if the other F K be bisected in H, by a straight line P H at right angles; the perpendicular P H will somewhere touch the section, provided, in the hyperbola, O K F be an obtuse angle. And, on the contrary, if P H touch the section and bisect F K in H at right angles, then O K will be equal to the transverse axis. This is evident from articles 101, 32, and 102. BOOK II.

105. The rest remaining as in the last article, let F N be at right angles to the straight line L N touching the ellipse or hyperbola in L, and let F N be produced to M, so that N M may be equal to F N; then a straight line, bisecting K M at right angles will pass through the focus O. For, by the last article, O M is equal to the transverse axis, and consequently equal to O K. If therefore O Q be drawn, bisecting K M in Q, the angles (8. i.) O Q K, O Q M will be equal. Hence, from this converse, the article is evident. Fig. 47, 48.

106. The rest remaining as in article 102, in K F let the points L, N be so taken that we may have the following proportions, Fig. 49, 50.

K L : F L :: A B : O F :: O K : O F :: K N : N F;

and then the circumference of a circle described on L N as a diameter will pass through O.

For in the ellipse let K O, but in the hyperbola let F O be produced to M, and let O L, O N be drawn. Then in the ellipse, the angles (3. vi.) K O F, F O M* are bisected by O L, O N; but in the hyperbola, the angles K O F, K O M* are bisected by O L, O N, and therefore (13. i.) in each section L O N is a right angle, and (31. iii.) the article is evident.

* That the angle F O M in the ellipse, and the angle K O M in the hyperbola, is bisected by O N, is proved by Simson, and also by Playfair, in their Prop. A, in the sixth Book of their edition of Euclid.

BOOK II.

Sir Isaac Newton makes much use of the properties expressed in the three last articles. See the Principia, sect. 4. book i.

PROP. XVI. Fig. 51, 52.

107. If a straight line (P H) touching an ellipse or hyperbola meet a straight line (F H) drawn from either of the foci, and be at right angles to it, the straight line (C H) joining the center and the point of concourse will be equal to the semitransverse axis: or, if a straight line touch an ellipse or hyperbola, and straight lines (P F, P O) be drawn from the point of contact to the foci, a straight line (C H) drawn from the center to the tangent, and parallel to (P O) either of the two drawn to the foci, will be equal to the semitransverse axis.

Part I. Let F H meet P O in K, and then as each of the angles F H P, K H P is a right one, and as, by article 103, the angles K P H, F P H are equal, F H (26. i.) is equal to H K, and P F to P K, and therefore, by article 96, O K is equal to A B. But as C O is equal to C F, and K H to H F, F C : F O : : F H : F K, and (2. vi.) O K, C H are parallel. Consequently, F O : F C : : O K or A B : C H, and therefore (15. v.) C H is equal to C A or C B.

If P H meet O G in G and be at right angles to it, it may be proved in the same way that the straight line C G is equal to C A or C B.

Part II. Let F H be drawn and meet P O in K, and then (2. vi.) F C : C O : : F H : H K, and therefore (14. v.) F H is equal to H K. But, by article 103, the angles F P H, K P H are equal, and therefore (3. vi.) F H : H K : : P F : P K, and P F is equal to P K, and by article 96, O K is equal to A B. Consequently, it being (4. vi.) F O : F C : : O K : C H, C H is equal to C A or C B.

If C G parallel to F P meet P H in G it may be proved in the same way that C G is equal to C A or C B. BOOK II.

108. The rest remaining as above, if the straight line L M drawn through C the center, and parallel to the tangent G P, meet P F in M and P O in L, the segments P M, P L are equal; and each of them is equal to A C or C B. For (34. i.) P M is equal to C G and P L is equal to C H.

The demonstrations of the 11th and 12th Propositions of the first Book of the Principia depend, in a very considerable degree, on this property.

109. The rectangle contained under two straight lines (O G, F H), drawn from the foci of an ellipse or hyperbola to a tangent (G P H), and at right angles to it, is equal to the square of the semiconjugate axis. And the rectangle contained under two straight lines (C K, M P), drawn from the transverse axis of an ellipse or hyperbola to a tangent, and at right angles to it, is equal to the square of the semiconjugate axis, if one of them be drawn from the center, and the other meet the tangent in the point of contact. PROP. XVII. Fig. 53, 54.

Part I. Let H C be drawn, and let it meet G O in Q. Then as F H, O G are at right angles to the tangent, they (28. i.) are parallel, and therefore (29. and 15. i.) the triangles H C F, Q C O are equiangular; and as, by article 89, C F is equal to C O, O Q (26. i.) is equal to F H and C Q to C H=C A=C B, by article 107. Consequently, if with C as a center, and C H or C Q as a distance, a circle be described, it will pass through A, B and also G, by article 107, and (35, and cor. 36. iii.) $GO \times HF = GO \times OQ = AO \times OB = CD^2$, by article 88.

Part II. Let P L be an ordinate to the conjugate axis, and let P N be an ordinate to A B, and let the conjugate axis meet the tangent in I. Then as, by ar-

BOOK II. ticle 66, P L is parallel to C N and P N to C L, P N (34. i.) is equal to C L. As C K, M P are at right angles to the tangent, they are (28. i.) parallel, and the angles P M N, K C N (29. i.) are equal; and, by article 59, each of the angles P N C, L C N is a right one. The triangles I C K, M P N are therefore equiangular, and (4. vi.)

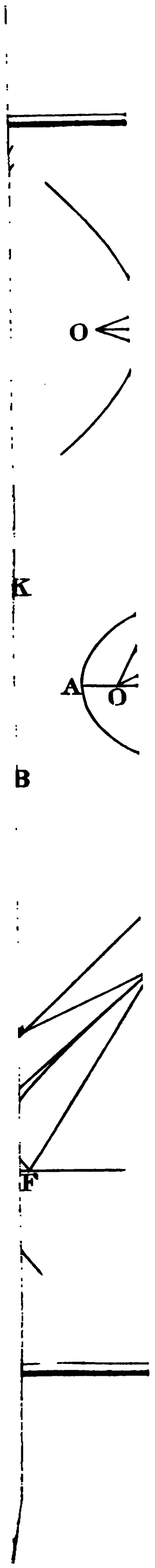
C K : C I :: P N or C L : P M.

Hence, (16. vi.) $CK \times PM = CI \times CL = CD^2$, by article 78, and (17. vi.).

PROP. XVIII. Fig. 55, 56. 110. If a straight line (G P H) touching an ellipse or hyperbola, be limited by tangents (A G, B H) passing through the vertices of the transverse axis, the circumference of a circle described about it as a diameter will pass through the foci; and the rectangle under the two straight lines (PO, P F) drawn from the point in which it touches the section to the foci will be equal to the square of (C L) the semidiameter parallel to it.

Part I. By articles 59 and 66, each of the angles G A B, H B A is a right one; and by articles 82, 88, $AG \times BH = AF \times BF$. Consequently, (16. vi.) B H : B F :: A F : A G; and therefore, the straight lines H F, G F being drawn, the angles (6. vi.) H F B, A G F are equal, as are also the angles B H F, A F G. The angles B F H, A F G together are therefore equal to a right angle, and consequently H F G is a right angle, and (31. iii.) this part is evident: for it may be proved in the same way, that straight lines drawn from H and G to O will contain a right angle.

Part II. The rest being as above, let the straight line F I be at right angles to G P H, and let it be produced and meet P O in K. Then, by article 103, the angles F P H, K P H are equal; and therefore (26. i.) P K is equal to P F, and K I to F I; and hence it is evident (3. iii.) that the circumference of the circle


O
K
A
O
B
F

GOFH passes through K. Consequently (35. and cor. 36. iii.) $OP \times PK = OP \times PF = GP \times PH = CL^2$, by article 82. BOOK II.

111. A straight line (F H) drawn from either of the foci of an ellipse or hyperbola, perpendicular to (G P H) a tangent, is to a straight line (F P) drawn from the same focus to the point of contact, as the semiconjugate axis to (C R) the semidiameter parallel to the tangent. PROP. XIX. Fig. 53, 54.

For let O G be perpendicular to the tangent, and let O P be drawn; and then as, by article 103, the angles F P H, O P G are equal, the triangles F P H, O P G are equiangular; and therefore we have the following proportions:

$$FH : FP :: OG : OP,$$

(16. v.) $FH : OG :: FP : OP$,

(22. vi.) $FH \times OG : FP \times OP :: FH^2 : FP^2$; and by articles 109, 110, $FH^2 : FP^2 :: CD^2 : CR^2$,

(22. vi.) $FH : FP :: CD : CR$.

112. Every thing remaining as in article 111,

$$FH : FP :: OG : OP :: CK : CH = CA = CB,$$

by art. 107, and therefore (11. v.)

$$CK : CB :: CD : CR.$$

113. By the last article (and 1. vi.)

$$CK \times PM : CB \times PM :: CD^2 : CR \times CD,$$

and therefore, by article 109,

(and 14. v.) $CB \times PM = CR \times CD$, and consequently

(16. vi.) $CB : CD :: CR : PM$.

114. If a straight line (I R) touch an ellipse or hyperbola (P B), and from the point of contact (P) two straight lines be drawn to an axis, the one (P H or P G) an ordinate to it, the other (P K or P M) perpendicular to the tangent, the segment (C H or C G) of the axis between the center and ordinate, will be to the segment (K H or G M) of the same axis between PROP. XX. Fig. 57, 58.

BOOK II. the perpendicular and ordinate, as the axis to its parameter.

Let the tangent meet the transverse axis in R and the conjugate axis in I. Then, by article 79,

$CH \times HR = AH \times HB$, and (cor. 8. and 17. vi.)

$KH \times HR = PH^2$, and (1. vi.)

$CH : KH :: CH \times HR : KH \times HR$.

Hence, $CH : KH :: AH \times HB : PH^2 :: AB :$ the parameter of A B, by article 73.

Fig. 57. Again, in the ellipse, by article 79,

$CG \times GI = EG \times GD$, and (cor. 8. and 17. vi.)

$GM \times GI = PG^2$, and (1. vi.)

$CG : GM :: CG \times GI : GM \times GI$.

Hence, $CG : GM :: EG \times GD : PG^2 :: DE :$ the parameter of D E, by article 73.

Fig. 58. Lastly, in the hyperbola, as, by article 66, P G is parallel to A B, and P H to D E,

(2. vi.) $CG : GM :: KP : PM :: KH : HC$,

and (1. vi.) $CG : GM :: KH \times RH : HC \times RH$.

But, as above,

$KH \times RH = PH^2$, and $HC \times RH = AH \times HB$.

Hence, (11. v.)

$CG : GM :: PH^2 : AH \times HB :: DE^2 : AB^2$, by article 67 (and 15. v.) But, by article 72, (and cor. 2. 20. vi.) $DE^2 : AB^2 :: DE :$ the parameter of D E, and therefore (11. v.) $CG : GM :: DE :$ the parameter of D E.

PROP. XXI. Fig. 57, 58. 115. If a straight line (I R) touch an ellipse or hyperbola, and a straight line (P M) be drawn from the point of contact at right angles to it, and meet the axes (in K, M), the rectangle under the segments (P M, P K) of the perpendicular, between the point of contact and the axes, will be equal to the square of the semidiameter (C L) parallel to the tangent.

For let the tangent meet the transverse axis in R and the conjugate in I, and then the triangles I P M, I C R, P K R being equiangular, BOOK II.

$$PM : PI :: PR : PK, \text{ and } (16.\text{ vi.})$$

$$PM \times PK = PI \times PR = CL^2, \text{ by article 83.}$$

116. By the last, and article 110,

$$PM \times PK = PF \times PO.$$

117. If a circle (A M B L) be described about the transverse axis of an ellipse as a diameter, and an ordinate (H I) to this axis be produced to the circumference (at K), the straight line (K I) thus produced will be to the ordinate as the transverse axis to the conjugate. For, by article 68, PROP. XXII. Fig. 59.

$$AI \times IB : HI^2 :: AC^2 : CD^2.$$

But, (35. iii.) $AI \times IB = KI^2$, and therefore

$$KI^2 : HI^2 :: AC^2 : CD^2,$$

and, (22. vi.) $KI : HI :: AC : CD :: AB : DE.$

118. If from a point (F) in the conjugate axis of an ellipse (A D B E) a straight line (F G), equal to the difference of the semiaxes, be drawn to a point (G) in the transverse axis, and be produced beyond the transverse axis, so that the part (G H) produced be equal to the semiconjugate axis, the extremity of the part produced will be in the curve of the section. Or, if from a point (F) in the conjugate axis of an ellipse a straight line (F G), equal to the sum of the semiaxes, be drawn to a point (G) in the transverse axis, and if this line be so cut that the segment (G H) between the transverse axis and the point of section be equal to the semiconjugate axis, the point of section will be in the curve. PROP. XXIII. Fig. 59. Fig. 60.

For through the center C draw the straight line C K parallel to F H. Through H draw the straight line H K parallel to D E, and let it meet C K in K, and A B in I. Then (34. i.) the straight line C K is equal to F H. But as F G is equal to the difference or sum of C B, C D, and as G H is equal to C D, the straight line Fig. 59, 60.

FH is equal to CB; and consequently CK is equal to CB. With C, therefore, as a center, and CB as a distance, let a circle be described, and it will pass through K. Again, on account of the similar triangles CKI, GHI, $CK^2 : GH^2 :: KI^2 : HI^2$; and therefore, on account of the equals, $CB^2 : CD^2 :: KI^2 : HI^2$. But (3. and 35. iii.) the square of KI is equal to the rectangle under AI, IB; and therefore $CB^2 : CD^2 :: AI \times IB : HI^2$. The point H is therefore in the curve, by article 68.

119. *Scholium.* The instrument called by some *the trammels*, and by others the *elliptic compasses*, used by cabinet-makers, &c. for describing the curves of ellipses, are constructed on the property demonstrated in this Proposition. As the trammels are in general use, it is needless to give a description of them in this place. Lathes for making picture-frames, and ornaments of an elliptical form, are constructed on the same property.

ELEMENTS

OF

CONIC SECTIONS.

BOOK III.

Of the Parabola, the Directrices of the Sections, the Asymptotes of the Hyperbola, Conjugate Hyperbolas, Hyperbolic Sectors and Trapezia. Also of the Areas of the Sections, and of Circles having the same Curvature with them in given Points.

Fig. 61.

120. THE section F D C being a parabola, and V B E its vertical plane, as in articles 20 and 21, any straight line, as D I, in the parabola parallel to V B, the side in which V B E touches the cone, is called a *Diameter* of the parabola.

121. From the last article (and 9. xi.) the diameters of a parabola are parallel to one another; and, by article 49, any straight line drawn in the plane of a parabola, parallel to a diameter, will meet the curve in one point, and in one point only, and, by the last article, it will itself be a diameter.

122. Any straight line in a parabola, not parallel to a diameter, will meet the curve in two points. For any straight line drawn through V, the vertex of the cone, and in the vertical plane, and not in the same direction with V B, will fall without the opposite cones; and by

BOOK III. the demonstration of the second part of article 37, one plane may be drawn through this straight line to touch the conical superficies, and cut the plane of the parabola. The intersection also of this plane with the plane of the parabola will touch the parabola, and any straight line in the parabola parallel to this tangent will meet the curve in two points, by article 38. Hence (16. xi.) this article is evident.

123. The point in which a diameter of a parabola meets the curve is called the *Vertex* of the diameter.

124. If a straight line terminated by the curve of a parabola be bisected by a diameter, it is called a *Double Ordinate* to that diameter; and its half is simply called an *Ordinate* to it.

125. The segment of a diameter between its vertex and an ordinate is called an *Absciss* of that diameter.

126. The diameter of a parabola, which cuts its ordinates at right angles, is called the *Axis* of the parabola.

Fig. 63, 64. 127 *Lemma*. If a straight line (G R) touch a circle (E P F in P) and two straight lines (E P, F P) cutting the circle pass through the point of contact, and meet (A B *) a straight line parallel to the tangent, the rectangle (E P × P C) under the segments of the one, between the point of contact and the circumference, and between the point of contact and straight line parallel to the tangent, will be equal to (F P × P D) the rectangle under the segments of the other, between the point of contact and circumference, and between the point of contact and straight line parallel to the tangent.

For E F being drawn, the angle E F P (32. iii.) is

* The straight line A B may be on either side of the tangent R G, and it is not necessary, upon being produced indefinitely, that it should meet the circumference of the circle.

equal to the angle R P E, which (29. i.) is equal to the angle P C D. The triangles E P F, D P C are therefore equiangular, and (4. vi.) E P : P F :: D P : P C. Consequently, (16. vi.) the rectangle under E P, P C is equal to the rectangle under F P, P D. BOOK III.

128. If A B cut the circle in B, and P B be drawn, it may be proved in the same way that the rectangle under F P, P D is equal to the square of P B. For the tangent R P being produced to G, and B F being drawn, the angle B P G (32. iii.) is equal to the angle B F P; and (29. i.) it is also equal to the angle D B P. The triangles B F P, D B P are therefore equiangular, and (4. vi.) F P : P B :: P B : P D, and (17. vi.) the rectangle under F P, P D is equal to the square of P B.

129. The rest remaining as above, if B G be drawn parallel to F P, B G is (34. i.) equal to P D, and therefore, by the above, $FP = \frac{PB^2}{BG}$. Fig. 64.

130. If each of two diameters of a parabola meet a straight line, and if each of these straight lines cut, or one of them cut and the other touch the parabola, and if these two straight lines be parallel; then the segment of the one diameter, between its vertex and the line which it meets, will be to the segment of the other between its vertex and the line which it meets, as the square of the line which meets the first mentioned diameter, if a tangent, or the rectangle under its segments, if a secant, to the square of the line which meets the other diameter, if a tangent, or the rectangle under its segments, if a secant. PROP. I.

Suppose D I, G K to be two diameters of a parabola, and let D be the vertex of the one, and G the vertex of the other. Let L P, M N be two parallel straight lines, and let D I meet L P in L, and G K meet M N in M, and let L P, M N either both cut, or Fig. 62.

BOOK III. one of them cut and the other touch the parabola; then DL is to GM as the square of LP, if a tangent, or the rectangle under its segments, if a secant, to the square of MN, if a tangent, or the rectangle under its segments, if a secant.

Fig. 61. For let FGDC be the parabola as formed in the cone, and DI, GK the diameters mentioned above. Let the parabola cut the plane of the base in the straight line FKIC, and let the vertical plane cut it in BE, VB being the side along which the vertical plane touches the cone. Then DI, GK are parallel to VB, by article 120. Through the parallels VB, DI let a plane pass, and let it cut the cone in the side VDA, and the base in BIA. Through the parallels VB, GK let a plane pass, and let it cut the cone in the side VGH, and the base in BKH. Then (4. vi. and 16. v.)

$$DI : VB :: AI : AB, \text{ and}$$

$$VB : GK :: HB : HK; \text{ and therefore,}$$

by article 44, (and cor. 1. vi.)

$$DI : GK :: AI \times HB : AB \times HK.$$

But, by article, 127, (and 16. vi.)

$$HB : AB :: BI : BK,$$

$$\text{and } AI : HK :: AI : HK,$$

and article 44,

$$AI \times HB : AB \times HK :: AI \times BI : BK \times HK;$$

and therefore, (11. v. and 35. iii.)

$$DI : GK :: AI \times BI : BK \times HK :: FI \times IC : FK \times KC.$$

Consequently, supposing the straight line FKIC to have the same situation in the parabola in Fig. 62, as in Fig. 61, and that LP, MN are parallel to the base of the cone, we have, by article 52, the following proportions, arranged for ex æquali;

$$DL \quad : \quad DI \quad : \quad GK \quad : \quad GM$$

$$\left.\begin{matrix} t.\ LP^2 \\ \text{or} \\ s.\ L.\ P^r \end{matrix}\right\} : FI \times IC : FK \times KC : \left\{\begin{matrix} t.\ MN^2 \\ \text{or} \\ s.\ MN^r \end{matrix}\right.$$

Hence, (22. v.) BOOK III.

D L : G M : : t. L P^2 or s. L P^r : t. M N^2 or s. M N^r.

And supposing L P, M N not to be parallel to the base of the cone, but that M R, L S are parallel to it, and that they are either both secants, or one of them a secant and the other a tangent to the parabola; then, by the above, and article 48,

D L : G M : : t. L S^2 or s. L S^r : t. M R^2 or s. M R^r : : t. L P^2 or s. L P^r : t. M N^2 or s. M N^r.

131. A diameter (B G) of a conic section (A B C) bisects any secant (A C) it meets in the same section or opposite hyperbola, parallel to a tangent (D B E) passing through its vertex (B); and ordinates to a diameter (B G) and a tangent passing through its vertex are parallel to one another. PROP. II. Fig. 65.

In the ellipse and hyperbola this has been proved in article 64; and in the parabola it may be proved in the following manner.

Part I. Through A, C, let A D, C E be drawn parallel to B G, and let A D meet the tangent in D, and C E meet it in E. Then, by article 121, A D, C E are diameters; and, by article 130,

A D : C E : : D B^2 : B E^2; and

(34. i.) as A D, C E are equal, D B = B E, and therefore A G = G C.

Part II. If it be possible, let the straight line H R in the parabola A B C be bisected by the diameter B G in N, and not be parallel to A C or D E.

Through R draw R M parallel to the diameter B G, and through H draw H L parallel to A C or D E, and let it meet B G in K, the curve again in L, and R M in M. Then as H R is bisected in N, and as K N, M R are parallel, (2. vi.) H N : N R : : H K : K M, and H K is equal to K M. But, by Part I. H K is equal to K L, and therefore K L is equal to K M; which is ab-

BOOK III. surd. Consequently ordinates to the diameter B G must be parallel to one another, and to the tangent D E, passing through the vertex.

132. From the last, and article 64, it is evident that if in any conic section a secant be an ordinate to a diameter, any other secant in the section or opposite hyperbola, and parallel to it, will be an ordinate to the same diameter.

133. From the last article it is evident, that if a straight line bisect two parallel secants in a conic section it will be a diameter.

134. From the above a method of finding a diameter of a given parabola is evident. For two parallel straight lines being drawn in the parabola, a straight line bisecting them, and any straight line parallel to it, will be a diameter.

135. The method of finding the axis of a parabola is also evident from article 126. For, having found a diameter of the parabola, by the preceding article, let a straight line at right angles to it be drawn within the parabola, and limited both ways by the curve. Then the diameter bisecting this straight line will be the axis; for, being parallel to the diameter first found, it will (29. i.) bisect the straight line in the section at right angles.

PROP. III. Fig. 65.

136. The abscisses (B G), (B K) of a diameter (B G) of a parabola are to one another as the squares of the corresponding ordinates (A G, H K); and a third proportional (P) to an absciss and its corresponding ordinate is of the same length, whatever absciss be taken.

Part I. This follows from articles 131 and 52.

Part II. By hypothesis,

$BG : AG :: AG : P$, and (17. vi.) $BG \times P = AG^2$.
But by Part I, $BG : BK :: AG^2 : HK^2$, and therefore, $BG : BK :: BG \times P : HK^2$. Hence,

(1. vi.) $BG \times P : BK \times P :: BG \times P : HK^2$,
and (14. v.) $BK \times P = HK^2$; and therefore,
(17. vi.) $BK : HK :: HK : P$.

137. A third proportional to an absciss of a diameter of a parabola, and the corresponding ordinate, is called the *Parameter*, or *Latus Rectum* of the diameter. The parameter of the axis is frequently called the *Principal Parameter*, or *Latus Rectum*.

138. *Scholium*. By article 136, the rectangle under the absciss B K and the parameter P is equal to the square of the corresponding ordinate H K; and on account of this equality Apollonius called the section a parabola.

From the property demonstrated in article 136, the parabola is frequently denoted by an algebraical equation, in the following manner. Put the parameter of the diameter $BK = p$, the absciss $BK = x$, and the ordinate $HK = y$. Then, from the above, $x : y :: y : p$, and $p\,x = y^2$.

139. If a diameter (A D or R M) of a parabola meet a tangent (B D) or secant (H L) to the section, the rectangle under its segment *, (A D or R M,) and the parameter of the diameter to whose ordinates the tangent or secant is parallel, will be equal to the square of the tangent or the rectangle under the segments of the secant. PROP. IV. Fig. 65.

For let B G be the diameter passing through the point of contact, and let P be its parameter, and A G an ordinate to it. Then D G is a parallelogram, and by article 135,
$BG \times P = AG^2$, and therefore (34. i.) $AD \times P = BD^2$.

Again, let the secant H M be parallel to the ordinate A G or tangent B D, and then, by article 130,

$$AD : RM :: BD^2 : HM \times ML, \text{ and}$$

* When a diameter meets a tangent or secant, its segment is its part between its vertex and the point of concourse.

BOOK III. (1. vi.) $AD \times P : RM \times P :: BD^2 : HM \times ML.$

Hence, by the above (and 14. v.)

$$RM \times P = HM \times ML.$$

Fig. 66. 140. If two straight lines A B, B C meeting one another, both touch or both cut, or one of them touch and the other cut a parabola, the square of A B, if a tangent, or the rectangle under its segments, if a secant, will be to the square of B C, if a tangent, or the rectangle under its segments, if a secant, as the parameter of the diameter to whose ordinates A B is parallel to the parameter of the diameter to whose ordinates B C is parallel.

For let P be the parameter of the diameter to whose ordinates A B is parallel, and p the parameter of the diameter to whose ordinates B C is parallel; and let B D be the segment of the diameter passing through B. Then by the last article, t. AB^2 or s. $AB^r = BD \times P$, and t. BC^2 or s. $BC^r = BD \times p$, and therefore (1. vi.) t. AB^2 or s. AB^r : t. BC^2 or s. BC^r :: $P : p$.

Fig. 57, 58, 67. 141. If a straight line P R, touching a conic section in P, meet any diameter A B in R, and P H be an ordinate to the diameter, and P K, at right angles to P R, meet the diameter in K; the segment H R is called *the Subtangent*, the straight line P K *the Normal*, and the segment K H the *Subnormal*.

PROP. V. 142. The segment (B R) of the subtangent between the tangent (P R) and the vertex (B) of the diameter (A B) is greater in the ellipse but less in the hyperbola than its segment (B H) between the vertex and ordinate, the diameter in the hyperbola being a transverse, but in the parabola the vertex bisects the subtangent.

For, in Figures 57, 58, let C be the center, and then, by article 78, C R : C B :: C B : C H. Hence, by conversion in the ellipse, but by inversion and conversion in the hyperbola, C R : B R :: C B : B H, and

therefore (14. v.) in Fig. 57, B R is greater than B H, BOOK III.
but in Fig. 58, B R is less than B H.

In Fig. 67, let P H meet the curve of the parabola again in G, and then P G will be bisected in H, P G being a double ordinate to A B. Let a diameter G L meet the tangent P R produced in L. Then as H R, G L are parallel, (4, and 2, vi.)

P H : P G :: H R : 2 H R or G L :: P R : 2 P R or P L; and therefore, by art. 130, PL^2 or (4, ii.) $4\,PR^2 : PR^2$ :: G L or 2 H R : B R. That is $4 : 1 :: 2\,HR : \frac{HR}{2} =$ B R.

143. If two straight lines touching a conic section, PROP. VI.
or opposite hyperbolas, meet one another, the diameter bisecting the line joining the points of contact will pass through the point of concourse.

In the ellipse, hyperbola, or opposite hyperbolas, this has been proved in article 80.

In the parabola P B G let the two straight lines P R, Fig. 67.
G R touch the section in the points P, G, and meet one another in R, and let the diameter A B bisect P G, the straight line joining the points of contact, in H; the diameter A B will pass through R.

For, as P G is bisected by the diameter A B, it is a double ordinate to it; and therefore, by article 142, if H B be produced, and meet the tangents, its segment between B the vertex and the tangent P R will be equal to its absciss B H, and its segment between B and the tangent G R will also be equal to B H. The diameter A B will therefore meet both the tangents P R, G R in the same point, and consequently will pass through R, the point of concourse.

144. If two straight lines touching a conic section or opposite hyperbolas, meet one another, a straight line passing through the point of concourse, and bisecting

BOOK III. the line joining the points of contact, will be a diameter. The truth of this is evident from the last and article 81.

145. From the above it is evident, that if P G be a double ordinate to A B, a diameter of any conic section P B G, and if P R, touching the section in P, meet the diameter in R, then if the straight line R G be drawn, it will touch the section in G.

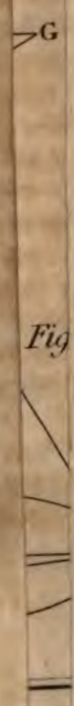
Fig. 68. 146. If from B, the vertex of the axis A B of the parabola P B M, a segment B F be taken in the axis equal to one fourth of the parameter of the axis, the point F is called the *Focus*, or *Umbilicus*, of the parabola.

147. The double ordinate T S, drawn through F, the focus of any conic section, is equal to the parameter of the axis passing through the focus. This has been proved in the ellipse and hyperbola in article 92. In the parabola the square of T F is equal to the rectangle under B F, and four times B F, by articles 136, 146, and therefore $4\ TF^2$ is equal to $4\ BF \times 4\ BF$. But (4. ii.) $4\ TF^2$ is equal to TS^2, and consequently T S is equal to 4 B F, and therefore, by the last article, equal to the parameter of the axis A B.

148. As in the ellipse and hyperbola, so in the parabola, the straight line touching the section in T, the extremity of the double ordinate drawn as above, is called the *Focal Tangent* to the parabola.

Fig. 68, 69, 70. 149. If the focal tangent T G, belonging to the focus F in any conic section P B M, meet the focal axis A B in X, the straight line X Y at right angles to A B is called a *Directrix* of the section. And if, in the ellipse or hyperbola, O be the other focus, and the focal tangent belonging to O meet the focal axis A B in K, the straight line K L at right angles to A B is also called a directrix of the ellipse or hyperbola.

150. As in the ellipse and hyperbola the foci F, O


G
Fig

are equally distant from C the center, it is evident from the last, and article 78, that the directrices X Y, K L are equally distant from the center. BOOK III.

151. In the parabola, the focus F and the directrix X Y are equally distant from B, the vertex of the axis, by the above and article 142.

152. If a tangent (B G) passing through the vertex (B) of the focal axis (A B) of a conic section (P T M) meet a focal tangent, its segment (B G) between the point of contact and point of concourse will be equal to the segment (B F) of the axis between the point of contact and the focus to which the focal tangent belongs. PROP. VII. Fig. 68, 69, 70.

As far as this relates to the ellipse, or hyperbola, it has been proved in article 94.

In the parabola, every thing remaining as in articles 146, 147, T F is double of F B, and therefore, by article 151, T F is equal to F X. Again, by article 131, T F, G B are parallel, and therefore (4. vi.) T F : F X :: G B : B X, and therefore G B is equal to B X, and consequently equal to F B. Fig. 68.

153. If a straight line (N R) drawn from a focal tangent (N T G X) and cutting the curve of a conic section (P B M in P) be at right angles to the focal axis (A B), and a straight line (P F) be drawn from the point in which it cuts the curve to the focus to which the tangent belongs, the two straight lines (N R, P F), thus drawn, will be equal to one another. PROP. VIII. Fig. 68, 69, 70.

For let N R produced meet the curve again in M, and let B G touch the section in B, and meet the focal tangent in G. Then, by article 131, B G is parallel to N M, and proceeding as in article 96,

by article 48, $TG^2 : TN^2 :: BG^2 : MN \times NP$,

(10. and 22. vi.) $TG^2 : TN^2 :: FB^2 : FR^2$,

(11. v.) $FB^2 : FR^2 :: BG^2 : MN \times NP$.

Consequently, by the last, (and 14. v.) $MN \times NP = FR^2$, BOOK III. and $MN \times NP + PR^2 = FR^2 + PR^2$, or (6. ii. and 47. i.) $NR^2 = PF^2$, and $NR = PF$.

154. By similar triangles, $NR : RX :: GB : BX$, and therefore, by the last and article 152, and (34. i.) if PY be perpendicular to the directrix XY, $PF : PY :: FB : BX$.

In the ellipse and hyperbola let KL be the other directrix, and let NV, PQ be perpendicular to it. Let the conjugate axis meet the tangent GT in I, and let KL meet it in L, and let PO be drawn to the other focus. Then, by article 98, CI is equal to CA or CB, and by similar triangles, $XC : XK :: CI : KL$; and as, by article 150, XK is double of XC, KL is double of CI, and therefore equal to AB. Consequently, by article 97, LV is equal to PO; and as by similar triangles, $LV : VN :: NR : RX$, it follows by the above, (and 34. i. and 11. v.) that $PF : PY :: PO : PQ :: FB : BX$.

155. By article 152, GB is equal to FB; and by article 98, CI is equal to CB; and therefore,

by article 154, $GB : BX :: PF : PY :: PO : PQ$,
(4. vi.) $GB : BX :: CB : CX$,
by article 78, $CB : CX :: CF : CB$.
Hence, (11. and 15. v.)

$$PF : PY :: PO : PQ :: FO : AB.$$

156. It is evident from the last article, that in the ellipse PF is less than PY, and PO less than PQ; but in the hyperbola PF is greater than PY, and PO greater than PQ.

Fig. 68. 157 A straight line PF drawn from any point P in the curve of a parabola to the focus, is equal to PY a straight line drawn from the same point perpendicular to the directrix. For, by article 154, $PF : PY :: FB : BX$; and, by article 142, FB is equal to BX.

PROP. IX. Fig. 72. 158. If from any point (P) in the curve of a parabola (P B R) a straight line (P F) be drawn to the focus (F), and a straight line (P D) perpendicular to the directrix (D X), the angle (D P F) contained by these straight lines will be bisected by a tangent (P E) passing through the same point.

For let the tangent P E meet the axis A B in E, and let P A be an ordinate to A B. Then, by article 142, A B is equal to B E; and as, by article 151, F B is equal to B X, A X is equal to F E, but A X = (34. i.) P D, and therefore, by article 157, P F = F E. Hence the angle F P E (vi. i.) = the angle F E P, and therefore (29. i.) the angle F P E = the angle D P E.

159. If a straight line P E touching a parabola meet the axis A B, the segment F E between the point of concourse and the focus is equal to the straight line P F drawn from the point of contact to the focus, as is evident from the last article.

160. A straight line (F G) drawn from the focus (F) of a parabola (P B R), perpendicular to a tangent (P E), is a mean proportional between the straight line (P F) drawn from the point of contact to the focus, and the segment (F B) of the axis between the focus and the vertex of the axis. PROP. X. Fig. 72.

For let the tangent P E meet the axis in E and draw B G, and let P A be an ordinate to the axis; and then, by the last article, P F, F E are equal, and therefore (5. i.) the angles F P E, F E P are equal. Consequently (26. i.) P G is equal to G E, and the angles P F G, E F G are equal. Hence, as by article 142, A B is equal to B E, P G : G E :: A B : B E;

and therefore (2. vi.) G B is parallel to P A and G B F is a right angle. Hence,

(4. vi.) P F : F G :: F G : F B.

This is Lemma XIV. Lib. I. of the Principia.

BOOK III. 161. By the last article, and (cor. 2, 20. vi.)

$$PF^2 : FG^2 :: PF : FB.$$

162. It follows from article 160, that the concourse of any tangent P E with a straight line F G, drawn from the focus of the parabola perpendicular to it, is in the straight line B G, which touches the parabola in the vertex of the axis; for it was there proved that G B is parallel to the ordinate P A.

163. If a straight line touch a parabola, and cut a straight line drawn from the focus to the directrix at right angles, it will bisect it. For the rest remaining as in the last three articles, let F G produced meet the directrix in D. Then as G B, D X are perpendicular to the axis, they are parallel, and

(2. vi.) F B : B X : : F G : G D; and therefore, by article 142, F G is equal to G D.

164. *Scholium*. If a straight line pass through a point moving in the curve of a conic section, and always touch the section, and if a straight line revolve about a focus of the section as a center, and be always perpendicular to the moving tangent, the magnitude of the perpendicular will be less varied in the hyperbola than in the parabola, but it will be more varied in the ellipse than in the parabola.

Fig. 72, 73, 74. For let the point P be supposed to move in the curve B P of the conic section P B R, and let the straight line P E accompany it in its motion, and always touch the section. Let the straight line F G revolve about F, a focus of the section, and let it be always perpendicular to the tangent P E.

Fig. 73, 74. In the ellipse and hyperbola let C be the center, C D the semiconjugate axis, and C H the semidiameter parallel to the tangent P E. Then, by article 111, F G : F P : : C D : C H, and therefore, (22. vi.) $FG^2 : FP^2 :: CD^2$:

CH^2. But O being the other focus, and PO being drawn, by article 110, $CH^2 = FP \times PO$, and therefore $FG^2 : FP^2 :: CD^2 : FP \times PO$, and $FG^2 = \frac{FP^2 \times CD^2}{FP \times PO} = \frac{FP \times CD^2}{PO}$. For the same reasons if p denote another position of the moving point, F g the perpendicular at that position, and F p, p O straight lines drawn from p to the foci, then $Fg^2 = \frac{Fp \times CD^2}{pO}$. Consequently $FG^2 : Fg^2 :: \frac{FP \times CD^2}{PO} : \frac{Fp \times CD^2}{pO} :: \frac{FP}{PO} : \frac{Fp}{pO}$.

If in the parabola p denote another position of the moving point, F g the perpendicular at that position, and p F a straight line drawn to the focus, then, by art. 160, (and 17. vi.) $Fg^2 = Fp \times FB$, and $FG^2 : Fg^2 :: FP \times FB : Fp \times FB :: FP : Fp$; or $FG^2 : Fg^2 :: \frac{FP}{FB} : \frac{Fp}{FB}$. Fig. 72.

In the hyperbola and ellipse, therefore, the square of the perpendicular FG varies as the value of the fraction $\frac{FP}{PO}$, and in the parabola as $\frac{FP}{FB}$.

But if the numerator and denominator of a fraction be each variable, then if they always increase or decrease in the same proportion, the value of the fraction will be always the same. Thus if the fraction be $\frac{Q}{R}$, and if while Q varies and becomes q, R varies and becomes r, and if $Q : R :: q : r$, then $\frac{Q}{R} = \frac{q}{r}$, by converting the proportion into an equation. From hence it is also evident, that the more nearly the nu-

BOOK III. merator and denominator increase or decrease in the same proportion, the less will the value of the fraction be varied; but, on the contrary, the more they differ from a proportional increase or decrease, the more will the value of the fraction be varied. Now in the hyperbola the difference between F P, P O is, in every situation of P, equal to A B the transverse axis, and therefore, in every instant they vary, they receive an equal increase or diminution; but however, in the parabola, F P may increase or diminish, F B remains constant. In the hyperbola therefore the value of the fraction $\frac{FP}{PO}$ varies less than the value of the fraction $\frac{FP}{FB}$ in the parabola. Again, in the ellipse the sum of P F, P O is equal to A B the transverse axis, and therefore if either of them increase, the other will diminish; and consequently in varying they will differ more from a proportional increase at the same time, or decrease at the same time, than F P, F B in the parabola. In the ellipse therefore the value of the fraction $\frac{FP}{PO}$ varies more than the value of the fraction $\frac{FP}{FB}$ in the parabola.

In the hyperbola therefore the square of F G varies less, but in the ellipse it varies more, than it varies in the parabola; and consequently F G varies less in the hyperbola, but more in the ellipse, than it varies in the parabola. See the Principia, Cor. 6. Prop. XVI. Lib. I.

PROP. XI. Fig. 72. 165. If from the point (P) in which the straight line (P E) touches the parabola (P B R) two straight lines (P A), (P H) be drawn to the axis, one of them (P A) an ordinate to it, the other (P H) at right angles to the tangent, the subnormal (H A) is equal to half the parameter of the axis. And if the tangent meet the axis in (E) the segment (E H) intercepted between the point of concourse and the normal (P H) will be equal to half the parameter of the diameter (K P D) passing through the point of contact. BOOK III.

Part I. Let p be equal to the parameter of the axis, and then (cor. 8. vi.) $EA \times AH = PA^2 = BA \times p$, by article 136, and therefore (16. vi.) $EA : BA :: p : AH$; and as, by article 142, B A is half of E A, the subnormal A H is half of p.

Part II. Let M be equal to the parameter of K P D, and then (cor. 8. vi.) $HE \times EA = PE^2 = BE \times M$, by article 139, and therefore (16. vi.) $EA : BE :: M : HE$; and as, by article 142, B E is half of E A, H E is half of M.

166. It is evident from the last article that the parameter of the axis is less than the parameter of any other diameter.

167. Every thing remaining in the Figure as stated in article 165, the normal P H is a mean proportional between half the parameter of the axis and half the parameter of the diameter K P D. For (cor. 8. vi.)

$$EH : PH :: PH : HA.$$

168. A straight line drawn from any point in the curve of a parabola to the focus is equal to a fourth part of the parameter of the diameter passing through the same point. If the point be the vertex of the axis, this is evident from article 146; but for any other point P let every thing remain as in the last Prop. and draw P F to the focus F. Then, by article 159, F P, F E are equal; and as E P H is a right angle, if with F as a center, and F P as a distance, a circle be described, it will pass (31. iii.) through E and H. Consequently F E, F P, F H are equal, and therefore, by Part II. of the last Proposition, each of them is equal

BOOK III. to a fourth part of the parameter of P K. This is Lemma XIII. Lib. I. of the Principia.

169. The distance of the vertex of any diameter of a parabola from the directrix is equal to a fourth part of the parameter of the diameter. This is evident from the last and article 157.

PROP. XII. Fig. 75. 170. A straight line (G K) drawn through the focus (F) of a parabola (P B K) and terminated both ways by the curve, is equal to the parameter of the diameter (P H), to which it is a double ordinate.

Let G K meet P H in H, let the tangent P E meet the axis A B in E, and let P F be drawn. Then, by articles 168, 159, P F is equal to F E, and each is one fourth of the parameter of the diameter P H. Consequently, as by articles 131, 121, P E is parallel to G K and P H to E F, (34. i.) P H is one fourth of the parameter of P H. Hence, by article 136, $PH \times 4PH = 4PH^2 = GH^2$, and $2PH = GH$, and therefore $4PH = GK$. The double ordinate G K is therefore equal to the parameter of P H.

171. If a straight line, as G K, in a parabola pass through the focus, and cut the diameter to which it is an ordinate, the absciss of the diameter will be equal to the distance of its vertex from the focus, and also from the directrix. For, as in the last article, $PH = PF = FE =$ the distance of P from the directrix by article 157.

Fig. 71. 172. If A M N be the vertical plane to the opposite hyperbolas E V F, G L H, as in article 26, and cut the cone in the sides A M, A N, and if planes A K, A I, touching the cone in the sides A N, A M, cut the plane of the hyperbolas in the straight lines K Q, I R, the straight lines K Q, I R are called the *Asymptotes* of either of the hyperbolas, or of the opposite hyperbolas.

173. As the vertical plane A M N is parallel to the plane of the hyperbolas E V F, G L H, the asymptotes K Q, I R (16. xi.) are parallel to A N, A M, sides of the cone.

174. The asymptotes K Q, I R do not meet the curve of either of the opposite hyperbolas. For the planes A K, A I touch the opposite cones in the sides A N, A M, and as the asymptotes are parallel to these sides, they do not meet either of the opposite conical superficies.

175. Any two straight lines T S, P O drawn in the planes A K, A I from the asymptotes to the sides A N, A M, and parallel to the base of the cone, are equal, and touch the conical superficies. For let N M, I K, N K, M I be the lines of common section of the base with the vertical plane, the plane of the hyperbolas, and the planes A K, A I. Then, as Q K, S N are parallel, and as S T, being parallel to the base, is parallel to N K, the straight line S T (34. i.) is equal to N K. For the same reasons O P is equal to M I. Also N K, M I are equal. For if N K, M I be parallel, then M K is a parallelogram, and (34. i.) N K is equal to M I. But if N K be not parallel to M I, let them meet in W; and as they are in the tangent planes, they will touch the circle M D N, the base of the cone. The straight lines W N, W M (36. iii.) are therefore equal, and (2. vi.) W N : N K : : W M : M I. Consequently (14. v.) N K is equal to M I, and therefore S T, O P are equal, and as they are in the planes A K, A I, they touch the conical superficies.

176. The angle I C K, or Q C R, within which either of the opposite hyperbolas is situated, is called the *interior angle of the asymptotes;* and the angle R C K, or Q C I, opposite to it, is called the *exterior angle of the asymptotes.*

BOOK III.

PROP. XIII.

Fig. 71.

177. Any straight line (L C V) passing through the point (C) in which the asymptotes meet, in the same plane with them, and falling within the interior angle of the asymptotes, will meet each of the opposite hyperbolas (E V K, G L H); and the point in which the asymptotes meet is the center of the opposite hyperbolas.

Part I. Every thing being in the Figure as stated in article 172, let A C be drawn, and through A C, L C V let a plane pass, and let its line of common section with the vertical plane be A Z. Then (16. xi.) A Z, C V are parallel, as are also A N, C K; and therefore (10. xi.) the angles Z A N, V C K are equal. In the same manner it may be demonstrated that the angles M A Z, I C V are equal, and that the angle M A N is equal to the angle I C K. The straight line A Z therefore, passing through A the vertex of the cone, falls within the opposite superficies, and consequently the plane passing through A Z, L C V must cut the opposite cones in two sides, one of them being on the right of the plane A M N, and the other on the left as viewed in the Figure. But as the planes A M N, Q K I R are parallel, and as L C V is parallel to A Z, the side of the cone on the right of A M N and the straight line L C V must meet the hyperbola E V F in the same point; and the side of the cone on the left of A M N and the straight line L C V must meet the hyperbola Q L R in the same point.

Fig. 78. Part II. Let A B D, E F be opposite hyperbolas, and let the asymptotes I P, R N cut one another in C. Let L M touching the hyperbola A B D in B meet the asymptotes in L, M, and let Q E parallel to it touch the opposite hyperbola in E, and meet the asymptote R N in Q. Let the secant A D be parallel to these tangents and meet the asymptotes in N, P. Let Q Z,

LX, MY, NV, PW, be straight lines parallel to the base of the cone in which the hyperbolas were formed, and let them be supposed to have touched the conical superficies or opposite superficies in the points Z, X, Y, V, W respectively. Then, by article 175, QZ, LX, MY, NV, PW are equal to one another, and therefore their squares are equal; and, by article 46, we have the following proportions:

$$ZQ^2 : QE^2 :: XL^2 : LB^2,$$
$$XL^2 : LB^2 :: YM^2 : MB^2,$$
$$VN^2 : DN \times NA :: WP^2 : AP \times PD.$$

Hence (14. v.) $QE^2 = LB^2$, $LB^2 = MB^2$, $DN \times NA = AP \times PD$, and therefore, $QE = LB$, $LB = MB$; and, by article 60, NA is equal to PD.

Let CB produced meet NP in O, and then, by similar triangles,

$CB : CO :: LB : NO :: MB : PO$; and therefore (14. v.) NO is equal to PO, and AD is bisected in O. Hence, by article 62, OC produced must pass through E; and, by similar triangles,

$LB : BC :: QE : EC$, and

(14. v.) BE is bisected in C. Hence, by articles 61 and 63, C is the center of the opposite hyperbolas.

178. It is evident from the last, and articles 61 and 63, that a straight line CT passing through C and falling within RCP or ICN, the exterior angle of the asymptotes, and parallel to the ordinates of EB or to the tangents passing through the vertices B, E is the conjugate diameter to EB.

179. If a straight line as LM touch an hyperbola and meet the asymptotes, its segments between the points of contact and the asymptotes will be equal. This is proved in the demonstration of article 177; and also, that if a straight line as AD cutting an hyperbola meet the asymptotes in N, P its segments NA, PD

BOOK III. between the curve and asymptotes are equal. The property is the same when any straight line as F H cutting the opposite hyperbolas in F, H, meets the asymptotes in Q, M. For, the rest remaining with respect to the Figure, by article 46,

$YM^2 : HM \times MF :: ZQ^2 : FQ \times QH$, and, therefore, (14. v.) by article 60, H M is equal to F Q.

180. If through any point P in either asymptote a straight line, as K G, be drawn, meeting the opposite hyperbolas in K and G, and a straight line, as P A, be drawn, meeting the curve of the hyperbola A B D in D and A; the rectangle under K P, P G will be equal to the square of the semidiameter parallel to K G, and the rectangle under A P, P D will be equal to the square of the semidiameter parallel to A P. For let E B be the diameter parallel to K G, the rest remaining as in the last three articles, and let the straight line C U be parallel to the base of the cone in which the section was formed, and let it be supposed to have touched the conical superficies in U. Then, by article 175, P W, C U are equal, and by article 46,

$PW^2 : KP \times PG :: CU^2 : CB^2$ or CE^2, and therefore (14. v.) $KP \times PG = CB^2$.

Again, by article 68,

$KP \times PG : AP \times PD :: CB^2$: the square of the semidiameter parallel to A P; and therefore (14. v.) $AP \times PD$ is equal to the square of the semidiameter parallel to A P.

181. A straight line as L M, touching the hyperbola A B D in B, and meeting the asymptotes in L, M, is parallel and equal to the second diameter conjugate to the transverse diameter E B, passing through the point of contact. For it may be proved, as in the last article, that the square of B M is equal to the square of the semidiameter parallel to it, and

therefore, by articles 179, 61, 63, the assertion is evidently true. BOOK III.

182. Either asymptote, and either of the opposite curves, being continually extended, will approach nearer and nearer to one another, but can never meet. For, the rest remaining, let C S be the semidiameter parallel to the tangent L M or the secant A D P, and then, by article 180, $CS^2 = AP \times PD$. Now, if we suppose the straight line A P continually to move from B, always to be parallel to C S, and cutting the curve in A, D, meet the asymptote in P, the equality of the fixed square CS^2 and the variable rectangle $AP \times PD$, will still hold. But, by such a motion, A P must continually increase, and therefore P D must continually decrease. The points P, D however cannot coincide; for, by article 174, the asymptote cannot meet the curve.

183. According as a transverse diameter (A B) of an hyperbola (G B) is greater, equal to, or less than its conjugate diameter (D E), the interior angle (K C I) of the asymptotes (C K, C I) is an acute, a right, or an obtuse angle; and any other transverse diameter (F G) is greater, equal to, or less than (H L) its conjugate diameter. PROP. XIV. Fig. 76.

For let N I, touching the hyperbola in B, the vertex of A B, meet the asymptotes in N, I; and let K M, touching the hyperbola in G, the vertex of F G, meet the asymptotes in K, M. Then, by articles 179, 181, N I is bisected in B, and K M in G; and N I is equal to D E, and K M equal to H L, and therefore, according as A B is greater, equal to, or less than D E, C B is greater, equal to, or less than B I. But if with B as a center and B I as a distance a circle be described, its circumference will pass through N, and according as C B is greater, equal to, or less than B I, it

BOOK III. will pass between C and B, through C, or on the opposite side of C from B. Consequently (by 31. iii. and 21. i.) according as C B is greater, equal to, or less than B I, the angle K C I is an acute, a right, or an obtuse angle. Again, if with G as a center and G M as a distance a circle be described, its circumference will pass through K, and (by 31. iii. and 21. i.) according as the angle K C I is an acute, a right, or an obtuse angle, the circumference of the circle will pass between C and G, through C, or on the opposite side of C from G. Consequently, according as K C I is an acute, a right, or an obtuse angle, C G is greater, equal to, or less than G M.

If two conjugate diameters of an hyperbola be equal, or if the angle contained by the asymptotes be a right one, it is called an *Equilateral Hyperbola*.

PROP. XV. Fig. 79. 184. The rectangle under the two straight lines (A E, A D) drawn from a point (A) in the curve of an hyperbola (A V) to the asymptotes (C H, C K) is equal to the rectangle under two straight lines (B F, B G) parallel to them, drawn from a point (B) in the curve of the same or opposite hyperbola to the asymptotes.

For draw A B, and let it meet the asymptotes in H and K, then as A E, B F are parallel, and as B G, A D are parallel,

(4. vi.) A E : B F : H A : H B;

and B G : A D : K B : K A. But, by article 179, H A is equal to K B, and therefore H B is equal to K A. Consequently,

(11. v.) A E : B F :: B G : A D, and therefore

(16. vi.) $AE \times AD = BF \times BG$.

185. If A E, B F be parallel to the asymptote C K, and A D, B G be parallel to the asymptote C H, the

parallelograms E D, F G (14. vi.) are equal. For the angle at C being common to the two parallelograms, they are equiangular, and, by the above, the sides round the equal angles are reciprocally proportional. BOOK III.

186. If from any two points as A, B in the curve of an hyperbola A V B two straight lines A E, B F, parallel to one of the asymptotes as C K, be drawn to the other asymptote C H; then C F : C E :: E A : F B. And the semitransverse diameters C A, C B being drawn, the triangles C F B, C E A are equal.

187. If C H, C K be the asymptotes of an hyperbola A V B, and if from any two points F, E in C H, straight lines F B, E A be drawn parallel to C K, and if F B be drawn to the curve and E A towards it, and if C E be to C F as F B to E A, the point A will also be in the curve.

188. If A B be a transverse diameter of the opposite hyperbolas A, B, and D E the second diameter conjugate to it, and if D E be a transverse diameter of the opposite hyperbolas D, E, and A B the second diameter conjugate to it; the hyperbolas D, E are called the *Conjugate Hyperbolas* to one or both of the opposite hyperbolas A, B, and, on the contrary, the hyperbolas A, B are called the *Conjugate Hyperbolas* to one or both of the opposite hyperbolas D, E. When all the four hyperbolas A, D, B, E are mentioned together, they are called *Conjugate Hyperbolas*. Fig. 80.

The point C, in which the diameters A B, D E, and the asymptotes cut one another is the common center of the conjugate hyperbolas.

189. If A M, B N touching the opposite hyperbolas in the vertices A, B meet the asymptotes in M, N, they are parallel to one another, and also to D E, by article 66; and, by article 181. each of them is equal to C E.

BOOK III. If, therefore, M N be drawn it will be (33. i.) parallel to A B, and as A B is bisected in C, M N (34. i.) will be bisected in E; and, by article 66, M N will touch the hyperbola in E, and M E, E N will be each equal to A C or C B.

190. The straight line D B, joining the vertices of the conjugate diameters A B, D E is parallel to C N, one of the asymptotes, and it is bisected by the other C M. For let C M meet D B in G, and the tangent N B in F, and then, as B N, D C are equal and parallel, D B, C N (33. i.) are also equal and parallel. Again, by similar triangles, F B : B G : : F N : N C, and as F B is half of F N, B G is half of C N or its equal D B.

191. *Lemma.* If a magnitude A be to a magnitude B as a magnitude C to a magnitude D; then A will be to B as the difference of the antecedents A, C to the difference of the consequents B, D.

For let A be greater than C, and consequently (14. v.) B greater than D. Then, by alternation, A : C : : B : D, and (17. v.) A – C : C : : B – D : D; and again, by alternation, C : D : : A – C : B – D. But, by hypothesis, A : B : : C : D, and therefore (11. v.) A : B : : A – C : B – D.

PROP. XVI. Fig. 80. 192. Asymptotes (C N, C M) to opposite hyperbolas (A, B), are also asymptotes to the hyperbolas (D, E) conjugate to them.

For the rest remaining as in articles 188, 189, from any point I in the asymptote C M, let the straight line I K be drawn parallel to A B or M N, and let it cut the hyperbola E in H, K and meet the diameter D E in V. From I let any other straight line I O be drawn, and let it cut the hyperbola E in L, P and meet the other asymptote in O. Let C T be the semidiameter parallel to I O. Then, by similar triangles (and 22. vi.) BOOK III.

$CE^2 : EM^2 :: CV^2 : VI^2$, and

by article 68, $CE^2 : CA^2 = EM^2 :: DV \times VE : HV^2$. Hence, by 191, (and 6. ii. and 11. v.) $CE^2 : CA^2 :: (CV^2 - DV \times VE =) CE^2 : (VI^2 - HV^2 = HI \times IK$, and therefore, (14. v.) $HI \times IK = CA^2$. Again, by article 68, $HI \times IK : LI \times IP :: CA^2 : CT^2$, and therefore, (14. v.) $LI \times IP = CT^2$. In the same way it may be proved that $PO \times OL = CT^2$; and therefore $LI \times IP = PO \times OL$; and consequently, by article 60, I L is equal to O P. For the same reasons, as in article 182, it therefore follows that C M, C N are asymptotes to the opposite hyperbolas D, E.

193. The rest remaining, let Q S, parallel to C T or I O, touch the hyperbola E in R and meet the asymptotes in Q, S; and then, as, by the last article, the segments I L, O P, of any parallel secant, are equal to one another, it follows that Q S must be bisected in R. It also follows, from the last article, that $QR^2 = RS^2 = CT^2$, and therefore that $QR = CT = RS$.

194. One of the asymptotes (as Q R) of an hyperbola (A or B) is parallel to, and the other (F P) bisects, a straight line (L H) joining the vertices (L, H) of any two conjugate diameters (L M, H K): and the vertices (H, K) of any second diameter (H K) are in the curve of the hyperbolas (D, E) conjugate to it. PROP. XVII. Fig. 81.

Part. I. Let the straight line F O touch the hyperbola B in L, and meet the asymptotes in F, O; and through the points F, H let the straight line F N be drawn, and let it meet the other asymptote in N. Then, by article 181, L O, H C are parallel and equal, and therefore (33. i.) L H is parallel to the asymptote Q R and equal to O C; and F L being, by article 179, equal to L O, is equal to C H. Consequently (29. and 26. i.) C I is equal to I F, and L H is bisected in I.

BOOK III.

Part II. The conjugate diameters A B, D E being as in article 188, let D B be drawn, and let it meet the asymptote F P in G. Then, by article 190, D B is parallel to the asymptote Q R, and it is bisected in G. Consequently, by article 186, (and 16. vi.) $CG \times GB = CI \times IL$, and therefore $CG \times GD = CI \times IH$; and, by article 187, the vertex H is in the conjugate hyperbola D. In the same way it may be proved that the other vertex K is in the conjugate hyperbola E.

195. From the last article it follows, that the second diameters of opposite hyperbolas are transverse diameters of hyperbolas conjugate to them.

196. It is also evident, from article 194, that tangents to the conjugate hyperbolas D, B, passing through the vertices L, H of any two conjugate diameters meet in the same point, as F, in one of the asymptotes.

Fig. 82. 197. If each of the two straight lines D F, D E, meeting one another in D, touch or cut, or one of them touch and the other cut an hyperbola or opposite hyperbolas; and if G K be parallel to D F and G H, meeting it in G, be parallel to D E, and if G K, G H each touch or cut, or one of them touch and the other cut a conjugate hyperbola or hyperbolas; the square of D F, if a tangent, or the rectangle under its segments, if a secant, will be to the square of D E, if a tangent, or the rectangle under its segments, if a secant, as the square of G K, if a tangent, or the rectangle under its segments, if a secant, to the square of G H, if a tangent, or the rectangle under its segments, if a secant.

For let C B be a semidiameter parallel to D F, and C A a semidiameter parallel to D E, and then, by articles 194, 68,

t. DF^2 or s. DF^r : t. DE^2 or s. DE^r :: $CB^2 : CA^2$, and

t. GK^2 or s. GK^r : t. GH^2 or s. GH^r :: $CB^2 : CA^2$.

Hence (11. v.) t. DF^2 or s. DF^r : t. DE^2 or s. DE^r :: t. GK^2 or s. GK^r : t. GH^2 or s. GH^r. BOOK III.

198. If from the vertices (F, I) of two conjugate diameters (F G, I H) of an ellipse (A D B E) or hyperbola (B F) straight lines (F K, I L) be drawn to a third diameter (A B) and parallel to its ordinates, in the ellipse the sum of the squares of the segments (C K, C L) between the center (C) and points of concourse will be equal to the square of half the third diameter (C B or C A); but in the hyperbola the difference of the squares of the segments (C K, C L) will be equal to the square of half the third diameter. PROP XVIII. Fig. 83, 84.

For let D E be the diameter conjugate to A B, and let the straight line M N, touching the section in F, meet A B in N and D E in M. Then, by article 66, F K, I L are parallel to D E, and M N is parallel to I H. Hence

(2. vi.) $FN : FM :: KN : KC$, and therefore,
(22. vi.) $FN^2 : KN^2 :: FN \times FM : KN \times KC$,
& (4. & 22. vi.) $FN^2 : KN^2 :: CI^2 : CL^2$. Hence,
(11. v.) $CI^2 : CL^2 :: FN \times FM : KN \times KC$.

But by article 84, $CI^2 = FN \times FM$, and therefore $CL^2 = KN \times KC = AK \times KB$, by article 79. Hence in the ellipse $CL^2 + CK^2 = AK \times KB + CK^2 =$ (5. ii.) CB^2: but in the hyperbola, $CK^2 - CL^2 = CK^2 - AK \times KB =$ (6. ii.) CB^2.

199. The rest remaining as in the last article, let F Q, I P be drawn parallel to A B, and let them meet D E in Q, P, and then, Q K, L P will be parallelograms; and, by article 48,

$$PI^2 : DP \times PE :: AK \times KB : FK^2 = CQ^2.$$

But (34. i.) $PI = CL$, and therefore, by the last article, $PI^2 = CL^2 = AK \times KB$, and consequently, (14. v.) $CQ^2 = DP \times PE$. Hence, in the ellipse, $CP^2 + CQ^2 = CP^2 +$

BOOK III. $DP \times PE = (5. ii.)\ CE^2 = CD^2$; but in the hyperbola $CP^2 - CQ^2 = CP^2 - DP \times PE = (6. ii.)\ CD^2 = CE^2$.

PROP. XIX. Fig. 83, 84. 200. In an ellipse the sum of the squares of any two conjugate diameters is equal to the sum of the squares of the axes; but in an hyperbola the difference of the squares of any two conjugate diameters is equal to the difference of the squares of the axes.

For the rest remaining, as in the last two articles, let A B, D E be the axes, and then the angles at K, L, Q, P will be each a right one. Hence, in the ellipse, as by article 198,

$$CB^2 = CK^2 + CL^2, \text{ and,}$$

by the last article, $CD^2 = KF^2 + LI^2$,
$CB^2 + CD^2 = CK^2 + KF^2 + CL^2 + LI^2 = (47. i.)\ CF^2 + CI^2$. Consequently (4. ii.) $AB^2 + DE^2 = GF^2 + HI^2$.

In the hyperbola, by article 198,

$$CB^2 = CK^2 - CL^2, \text{ and}$$

by the last article, $CD^2 = CP^2 - CQ^2$, and therefore, $CB^2 - CD^2 = CK^2 - CL^2 - CP^2 + CQ^2 = CK^2 + CQ^2 - (CL^2 + CP^2) = (34. \text{ and } 47. i.)\ CF^2 - CI^2$. Consequently $AB^2 - DE^2 = (4. ii.)\ GF^2 - HI^2$.

PROP. XX. Fig. 85, 86. 201. A quadrilateral figure (M Q T N) whose sides pass through the vertices (P, F, G, R) of any two conjugate diameters (P G, F R) of an ellipse (A D B E), or conjugate hyperbolas (A, D, B, E), and touch the ellipse or hyperbolas, is a parallelogram, and is equal to the rectangle under the axes (A B, D E) of the ellipse or hyperbolas.

From C the center let C K be drawn perpendicular to M Q. By article 66, M Q, R F, N T are parallel to one another, and M N, P G, Q T are also parallel to one another. The quadrilateral figure M Q T N is therefore a parallelogram; and, as R F, P G are each bisected in C, the center, the parallelograms M C, Q C, T C, N C (36. i.) are equal, and therefore each of them is one fourth of M Q T N. As C K is at right angles to M Q, and as M Q, R F are parallel, C K (29. i.) is also at right angles to R C; and therefore, the rectangle under C K, C R (35. i.) is equal to the parallelogram M C. But, by article 112,

C K : C B :: C D : C R, and therefore,

(16. vi.) C K × C R = C B × C D, and 4 C K × C R = the parallelogram M Q T N = 4 C B × C D = A B × D E.

202. All parallelograms contained under tangents, passing through vertices of conjugate diameters of an ellipse, or conjugate hyperbolas, are equal to one another, and each of them is equal to the rectangle under the axes.

The twelfth Lemma of Sir Isaac Newton's Principia, Lib. I. and the tenth Proposition, depending upon it, are evident from the above.

203. If from C the center of the hyperbola M N Q any two semitransverse diameters C N, C Q be drawn to the curve, the figure C N Q, bounded by the semidiameters and the curve N Q, is called a *Hyperbolic Sector.* Fig. 77.

204. If C R, C G be the asymptotes of the hyperbola M N Q, and from any two points N, Q in the curve, straight lines N B, Q A parallel to the asymptote C R be drawn to the other asymptote C G, the figure A B N Q, bounded by the straight lines N B, B A, A Q and the curve N Q, is called a *Hyperbolic Trapezium.*

205. A hyperbolic sector C N Q and trapezium A B N Q upon the same portion of the curve are equal. For let C N cut A Q in P, and then, as by article 186, the triangles C A Q, C B N are equal, the rectilineal trapezium A B N P is equal to the triangle C P Q. To these equals add the figure P N Q, and then the hyperbolic

BOOK III. sector C N Q is equal to the hyperbolic trapezium A B N Q.

206. Any segment as C A, intercepted between C the center and A a point in either asymptote, is called an *Asymptotic Segment*, and the point A is called its extremity.

207. Any straight line drawn from the curve of an hyperbola, parallel to one asymptote, and meeting the other, is called an *Asymptotic Secant*.

PROP. XXI. Fig. 77. 208. If from the points (Q, M) in which a straight line (M Q) cuts, and the point (N) in which a straight line (G R) parallel to it touches an hyperbola (M N Q), straight lines (M D, Q A, N B) parallel to one of the asymptotes (C T) be drawn to the other (C X), they will cut off from the center proportional asymptotic segments (C A, C B, C D); and, conversely, if from the extremities of three proportional asymptotic segments (C A, C B, C D) straight lines (A Q, B N, D M) parallel to the other asymptote be drawn to the curve of the hyperbola, the straight line (M Q) joining the extreme points in the curve will be parallel to the tangent (G R) passing through the middle point in the curve.

Part I. Let the secant M Q meet the asymptotes in H, K, and let the tangent meet them in G, R. Then, on account of the parallel lines,

(10. vi.) H M : H D :: K Q : C A,
and G N : G B :: R N : C B.

But, by article 179, H M is equal to K Q, and G N to R N, and therefore (14. v.) H D is equal to C A, and G B to C B. Consequently, (7. v. and 4. vi.)

C A : C B :: H D : B G :: D M : B N, and
by article 184. D M : B N :: C B : C D.
Hence (11. v.) C A : C B :: C B : C D.

Part II. Every thing being, as already stated, with respect to the parallelism of the lines, and the points in which they meet the curve and the asymptotes, it may be demonstrated, as in the preceding part, that C A is equal to D H, and C B to B G. Consequently, on account of this equality,

C A : C B :: H D : B G. But,
by hypothesis, C A : C B :: C B : C D, and
by article 184, C B : C D :: D M : B N. Consequently, (11. v.) H D : B G :: D M : B N, and
by alternation, H D : D M :: B G : B N.

Hence, as D M, B N are parallel, the angles (6. vi.) D H M, B G N are equal, and (29. i.) H K, G R are parallel.

209. The rest remaining, if the secant L F be parallel to M Q, and consequently to G R, and F E, L O parallel to C T be drawn to C X; then, by Part I. in the last article, (and 17. vi.) $CE \times CO = CB^2 = CA \times CD$, and therefore (16. vi.) C E : C A :: C D : C O.

And conversely, the rest remaining as in the last article, if from the extremities F, L of the secant F L straight lines F E, L O parallel to C T be drawn to C X, and the hypothesis be C E : C A :: C D : C O, then L F will be parallel to M Q. For (16. vi.) $CE \times CO = CA \times CD = CB^2$; and it may be proved, as in the second part of the last article, that L F is parallel to G R, and consequently (30. i.) to M Q.

210. If from the extremities (E, A, D, O) of four proportional asymptotic segments (C E, C A, C D, C O), asymptotic secants (E F, A Q, D M, O L), be drawn to the curve of the hyperbola (M N Q), the hyperbolic trapezium (E A Q F), between the first and second secant, will be equal to the hyperbolic trapezium (D O L M) between the third and fourth. PROP. XXII. Fig. 77.

For M Q, L F being drawn, they will be parallel, by article 209. Draw C L, C M, C Q, C F. Let C V be the diameter to which the parallels M Q, L F are double ordinates, and let it meet M Q in T, and L F in V. Then (38. i.) the triangle C L V is equal to the triangle C F V, and the triangle C M T is equal to the triangle C Q T; and as the diameter C V bisects all straight lines parallel to M Q and terminated by the curve, the space T M L V is equal to the space T Q F V. Consequently (axiom 3. i.) the hyperbolic sector C F Q is equal to the hyperbolic sector C M L; and therefore, by article 205, the hyperbolic trapezia E A Q F, D O L M are equal.

211. The rest remaining as above, let C A, C B, C D, C X, &c. be a series of asymptotic segments in geometrical progression, and let the asymptotic secants A Q, B N, D M, X Y, &c. be drawn to the curve; the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progression.

For let G R touch the hyperbola in N, and then, by article 208, it is parallel to M Q. Let the diameter C T pass through N, and then, by articles 61, 63, it bisects M Q in T; and (38. i.) the triangles C T Q, C T M are equal. And as C T bisects every straight line parallel to M Q, and terminated by the curve, the space N M T is equal to the space N Q T. Consequently (axiom 3. i.) the hyperbolic sectors C N Q, C N M are equal; and therefore, by article 205, the hyperbolic trapezia A B N Q, B D M N are equal. As, by hypothesis, C B is to C D as C D to C X, it may be proved, in the same way, that the hyperbolic trapezia B D M N, D X Y M are equal; and the same mode of proof may be extended to any number of terms. Consequently the hyperbolic trapezia A B N Q, A D M Q, A X Y Q, &c. are in arithmetical progression *. BOOK III.

212. If a circle (A F B G) be described about the transverse axis (A B) of an ellipse (A D B E) as a diameter, the area of the circle will be to that of the ellipse as the transverse to (D E) the conjugate axis. PROP. XXIII. Fig. 87.

Let C be their common center, and let D E produced meet the circumference of the circle in F, G. Let a straight line of indefinite length be supposed to move with one of its extremities P in A C from A towards C, and during this motion let it be always perpendicular to A B, and let it cut the circumference of the circle in K and the curve of the ellipse in L. Then, by article 117, C F : C D :: P K : P L, and as the superficies passed over by P K, P L, must be to one another in the same proportion, the area A P K A : the area A P L A :: C F : C D. Consequently, when the moving line arrives at C F and coincides with it,

the area A C F A : the area A C D A :: C F : C D.

But the area A C F A is one fourth of the circle, and the area A C D A is one fourth of the ellipse, and therefore (15. v.) the circle A F B G : the ellipse A D B E :: C F : C D :: A B : D E.

213. An ellipse is equal to a circle, whose diameter is a mean proportional between its axes. For the rest

* As the hyperbola and its asymptotes may be indefinitely extended, it is evident that a series of asymptotic segments in geometrical progression, and a corresponding series of hyperbolic trapezia in arithmetical progression, may be continued to any assigned number of terms. From the nature of logarithms, therefore, the series of asymptotic segments C A, C B, C D, &c. as above, is analogous to a series of natural numbers in geometrical progression, and the series of hyperbolic trapezia A B N Q, A D M Q, &c. as above, is analogous to the logarithms of these natural numbers. To enter into an explanation of these analogies would be incompatible with the design of this work. It would also be needless, as books on the subject are well known, and in common use.

BOOK III. Fig. 87, 88. remaining, let M N be a mean proportional between A B, D E, and let it be the diameter of the circle M H N. Then, by article 212, (and 1. vi.) the circle A F B G : the ellipse A D B E :: A B : D E :: AB^2 : $AB \times DE = MN^2$, (by 17. vi.); and therefore (2. xii.) the circle A F B G : ellipse A D B E :: the circle A F B G : the circle M H N, and (14. v.) the ellipse A D B E is equal to the circle M H N.

214. From the last article (and 2. xii.) it is evident, that the areas of two ellipses are to one another as the rectangles under their axes.

The Cor. to Prop. XIV. Lib. I. of the Principia depends, in a great degree, upon this truth.

215. If T, *t* denote the transverse axes of two ellipses, and C, *c* their conjugate axes, then, by the last article, the one ellipse : the other ellipse :: $T \times C$: $t \times c$; and therefore, if T be equal to *t*, they are to one another (cor. 1. vi.) as C to *c*; but if C, *c* be equal, they are to one another as T to *t*.

PROP. XXIV. Fig. 89. 216. A triangle (F K L) contained under a double ordinate (K L to B H) in a parabola (K B L) and tangents (F K, F L) passing through its extremities, is to the parabolic area, contained by the curve (K B L) and the double ordinate, as 3 to 2.

For let G I be a tangent to the parabola at B the vertex of the diameter B H. Let it be supposed that a straight line of indefinite length moves with one of its extremities in B H, from B towards H; that during this motion it is always parallel to G I, and cuts the curve in K, and that a straight line K F accompanying K H is always a tangent to the curve at K and meets the diameter H B in F. Now, by these motions of K H, K F, the triangle K H F is increased on the one side of B by the motion of K H, and on the other by the motion of K F; for, by article 142, B H, B F are


C
E
A
B
P
T
ig.77
I
X
L
V
N
A

always equal to one another; but the parabolic area B K H B is increased by the motion of K H only. It therefore follows, that the velocity with which the triangle K H F increases is to the velocity, with which the parabolic area B K H B increases as the sum of the two velocities by which the triangle F K H increases, to the single velocity by which the parabolic area B K H B increases. Now, supposing K H to increase its distance at any time from B by an indefinitely small space, F will increase its distance from B by an indefinitely small and equal space, by article 142. But the space passed over by K H, being indefinitely small, is as a parallelogram, while the space passed over by the tangent K F is as a triangle, of the same altitude with the parallelogram. The increase, therefore, (41. i. and cor. 1. vi.) by the moving tangent K F is to the increase by the moving ordinate K H as 1 to 2. Consequently, the triangle F K H is always to the parabolic area B K H B as 3 to 2.

If the ordinate K H produced always meet the curve again in L, and if the tangent L I always accompany this extremity, by article 143, this tangent will always meet the other in F; and, for the same reasons as above, the triangle F L H will always be to the parabolic area B L H B as 3 to 2. Hence 3 : 2 : : the triangle F K H : the area B K H B : : the triangle F L H : the area B L H B, and therefore (12. v.) the triangle F K L : the parabolic area K B L K : : 3 : 2.

217. Let G I meet the tangents F K, F L in G, I and the diameters K M, L N in M, N. Then, by articles 66, 121, H M is a parallelogram, and (34. i.) K M = H B = B F, by article 142. The angles K M G, M G K (29, and 15. i.) are also equal to the angles F B G, B G F, and therefore, (26. i.) M G is equal to G B, and (4. i.) the triangles M G K, B G F are equal.

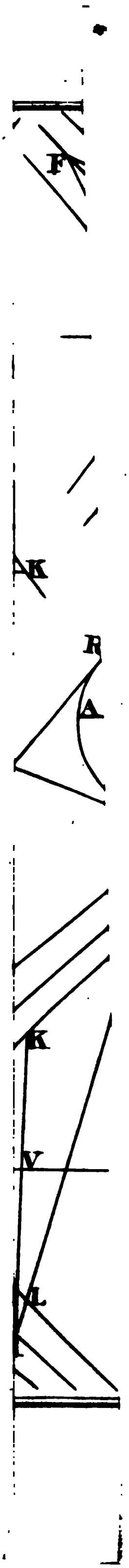
BOOK III. In the same way it may be proved that the triangles N I L, B I F are equal, and therefore the parallelogram M K L N circumscribed about the parabolic segment K B L is equal to the triangle K F L. Hence, by the last article, the circumscribed parallelogram is to the parabolic segment as 3 to 2, and therefore the parabolic segment K B L is two thirds of the circumscribed parallelogram K M N L.

218. As, by articles 216, 217, the parabolic area H B L H is two thirds of the parallelogram H N, the curvilinear area B L N B is one third of H N; and the straight line B L being drawn, the triangle (34. i.) H B L is one half of H N. Hence, $\frac{2\,HN}{3} - \frac{HN}{2} = \frac{4\,HN - 3\,HN}{6} = \frac{HN}{6}$ = the area contained by the straight line B L and the curve B L, and therefore this area is equal to half the area contained by the straight lines B N, N L and the curve B L. This curvilinear area B N L therefore is equal to two thirds of the triangle B N L, and the parabolic segment contained by the straight line B L and the curve B L is one third of the same triangle. See the Principia, Cor. 5. Lemma XI.

219. The *Curvature of a conic section, at any point,* is the angle contained by the curve and a tangent to it at that point.

Fig. 90. 220. *Lemma.* The curvatures of any two circles (B D G, B F H) are inversely as their diameters.

For let one touch the other internally at B, and let the straight line A B C be a common tangent to them at that point. Let the straight line B K be at right angles to A C and let it cut the circle B F H in H and B D G in K, and then (19. iii.) B H, B K are the diameters. In the tangent let any point C be


F
K
R
A
K
V
L

taken, and let a straight line C G be drawn cutting the outer circle in D, G, and the inner in E, F, and then (36. iii.) $GC \times CD = BC^2 = FC \times CE$; and therefore,

$$GC : FC :: CE : CD.$$

Let it now be supposed that a straight line of indefinite length moves with one of its extremities in C B from C towards B; that it is always parallel to C G, and cuts the outer circle in M and the inner in L, and then, as in the preceding proportion, M B will be to L B as its segment between the tangent and the inner circle, to its segment between the tangent and the outer circle, for these segments are analogous to C E, C D. Now, when the extremity of the moving line in C B falls into B, the segments analogous to C E, C D no longer exist, but their ratio with which they sunk into B is still that of B M to B L, and this ratio expresses the ratio of the magnitudes of the angles E B C, D B C, or that of the curvatures of the circles.

Let L H, M K be drawn, and then, as the angle M B K is common to the two triangles M B K, L B H, and as each of the angles B M K, B L H, (31. iii.) is a right one, M B : B L :: K B : B H :: the curvature of B E F : the curvature of B D G.

221. *Scholium.* As every point in the circumference of a circle is equally distant from the center, its curvature, or the deflection of the circumference from a tangent, is the same for every point in the circumference; but this is not the case in any one of the three conic sections. In each of them the curvature is evidently the greatest at a vertex of the focal axis. Proceeding from this point round an ellipse, the curvature continually decreases to a vertex of the conjugate axis. It then increases to the next vertex of the transverse axis; and a similar variation takes place in the other half of the elliptical curve. In an hyperbola and parabola the

BOOK III. curvature continually decreases as a point moving in the curve increases its distance, on either side, from the vertex of a focal axis. Throughout all this variation, however, a circle of corresponding curvature may be obtained; for, by the last article, the curvature of a circle decreases as its diameter increases, and consequently, it is only necessary to regulate the magnitude of the circle by the properties of the section with reference to the point at which the curvature is estimated.

PROP. XXV. Fig. 91. 222. Let E B F be a conic section, and A, B, C three points in the curve, and through A, B, C let the circle A B C D be described; then, if the extreme points, A C continually approach to B, and ultimately fall into it, the circle A B C D will ultimately have the same curvature with the section at B, or be the *Osculating Circle* for that point.

For as only one circle can pass through the three points (10. iii.), and as A, C continually approach to B, and ultimately fall into it, it is evident that no circle can be drawn through B so as to pass between the curve E B F and the circle A B C D.

It is manifest from the above, that the curve and circle may have the same tangent in the point B.

PROP. XXVI. Fig. 92, 93, 94. 223. An osculating circle (P E L) to any point (P) in a conic section (P G), will cut off from the diameter (P H) of the section passing through that point, a segment (P L) equal to its parameter.

Fig. 92, 93. First, let the section P G be an ellipse or hyperbola of which C is the center. Let the straight line P T be the common tangent to the section and circle in the point P, and let the straight line T F parallel to P H meet the tangent in T, the ellipse or opposite hyperbolas in G, F and the circle in D, E. Let the diameter C B be parallel to P T. Then, by article 68,

$CP^2 : CB^2 :: FT \times TG : PT^2$ or (36. iii.) its equal

$E T \times T D$. But if T F be alway parallel to P H, and move towards it and ultimately fall into it, when T falls into P, E into L, and F into H, it is evident, from article 222, that T G, T D must be equal. The above proportion, therefore, (cor. 1. vi.) then becomes $CP^2 : CB^2 :: PH$ or $2 CP : PL$, and $PL = \frac{2 CB^2}{CP}$. Again, by article 72, BOOK III.

$2 CP : 2 CB :: 2 CB : \frac{2 CB^2}{CP}$ = the parameter of P H, and therefore P L is equal to the parameter of P H.

Secondly, let the section P G be a parabola, and let the straight line P T be the common tangent to the parabola and circle in the point P, and let the straight line X be equal to the parameter of P H. Let the straight line T E parallel to P H, meet the tangent in T, the curve of the parabola in G and the circle in D, E. Then, by article 139, $G T \times X = P T^2 =$ (36. iii.) $E T \times T D$. But if T E be always parallel to P H, and move up to it, when T falls into P, and E into L, it is evident, from article 222, that G T, T D must be equal to one another. The above equation, therefore, then becomes $X = P L$, the parameter of the diameter P H. Fig. 94.

224. The rest remaining in each of the three sections, if the straight line L N at right angles to P L, and the straight line P N at right angles to P T, meet in the point N, P N will be the diameter of the osculating circle at P. For P N (19. iii.) must pass through the center, and (31. iii.) the straight line drawn from L to the other extremity of the diameter must contain a right angle with P L. Fig. 92, 93, 94.

225. If through O, the focus of the parabola, the straight line P R be drawn, and meet the circle of curvature again in R, P R will be equal to P L, the parameter of the diameter P H. Fig. 94.

BOOK III.

For let O V be parallel to T P and meet P H in V. Then, by article 171, P V is equal to P O; and, by article 127, $PL \times PV = PR \times PO$. Consequently $PR = PL$.

226. The chord of curvature P R, passing through the focus, for any point P in the parabola, varies as P O the distance from the focus. For, by the last article, P R is equal to the parameter, and therefore, by article 168, equal to four times P O.

Fig. 92, 93. 227. The rest remaining, in the ellipse and hyperbola, if the straight line P R, drawn through the focus O, meet the circle again in R, and C A be the semitransverse axis, then P R will be to the diameter C B as the semidiameter C B to C A. For, as in the demonstration of article 223, $2\,CB^2 = CP \times PL = PK \times PR$, by article 127. But, by article 108, $PK = CA$, and therefore $CA \times PR = 2\,CB^2$, and consequently $PR : 2\,CB :: CB : CA$.

228. By the last article (and 16. vi.) the chord of curvature P R passing through the focus, for any point P in the curve of an ellipse or hyperbola, varies as $\frac{2\,CB^2}{CA}$, or as CB^2, for $\frac{2}{CA}$ is a constant quantity.

229. Let PN, the diameter of the circle of curvature, or PN produced, meet C B in Q, and then, by article 127, $CP \times PL = PQ \times PN$. But $PL = \frac{2\,CB^2}{CP}$, and therefore $PQ \times PN = 2\,CB^2$ and $PN = \frac{2\,CB^2}{PQ}$.

Again, as C B is parallel to P T, and P N perpendicular to P T, P Q is equal to a perpendicular let fall from C on P T, and therefore, by article 112, if C M be the semiconjugate axis, $PQ : AC :: CM : CB$, and (16. vi.) $PQ \times CB = AC \times CM$, or $PQ = \frac{AC \times CM}{CB}$.

Consequently $PN = 2\,CB^2 \times \frac{CB}{AC \times CM} = \frac{2\,CB^3}{AC \times CM}$. BOOK III.

According to this expression, the diameter of the circle of curvature varies as CB^3, for $\frac{2}{AC \times CM}$ is a constant quantity; and, according to the former expression $\frac{2\,CB^2}{PQ}$, the same diameter of curvature varies as $\frac{CB^2}{PQ}$.

230. If in each of the sections I be the point in which PN meets the focal axis, PN the diameter of the circle of curvature will vary as PI^3, the cube of the normal.

For first, in the ellipse and hyperbola, by article 112, Fig. 92, 93. $CB = \frac{CA \times CM}{PQ}$, and therefore $2\,CB^2 = \frac{2\,CA^2 \times CM^2}{PQ^2}$. But, as in article 229, $PN = \frac{2\,CB^2}{PQ}$, and therefore $PN = \frac{2\,CA^2 \times CM^2}{PQ^3}$. Again, by article 109, $PI \times PQ = CM^2$, and therefore $PQ = \frac{CM^2}{PI}$, and $PQ^3 = \frac{CM^6}{PI^3}$. Consequently $PN = 2\,CA^2 \times CM^2 \times \frac{PI^3}{CM^6} = \frac{2\,CA^2 \times PI^3}{CM^4}$, and, as $\frac{2\,CA^2}{CM^4}$ is a constant quantity, PN varies as PI^3.

In the parabola, let OS be perpendicular to the tangent PT, and let NR be drawn. Then, OS, NP are parallel, each being perpendicular to the tangent, and therefore (29. i.) the angles SOP, RPN are equal, and each of the angles at S and R (31. iii.) is a right one, and Fig. 94.

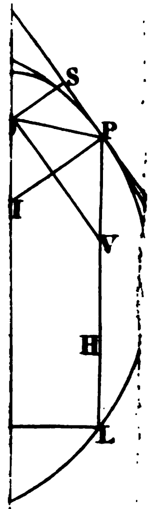
$$SO : PO :: PR \text{ or } 4\,PO : PN = \frac{4\,PO^2}{SO}.$$

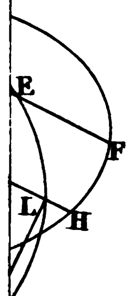
H

BOOK III.

But $PO=\frac{SO^2}{\frac{1}{4}L}=\frac{4\,SO^2}{L}$, by article 160, and putting $L=$ the parameter of the axis. Hence $PN=\frac{64\,SO^4}{L^2}\times\frac{1}{SO}=\frac{64\,SO^3}{L^2}$. Again, $2\,SO=PI$, as is evident from article 168, and therefore $8\,SO^3=PI^3$, and $PN=\frac{8\,PI^3}{L^2}$, and therefore, as $\frac{8}{L^2}$ is a constant quantity, PN varies as PI^3.

APPENDIX.


S
P
I
V
H
L


E
F
L
H

APPENDIX.

231. IF a straight line (P Y) touch a conic section (A P), and from the point of contact (P) two straight lines (P F, P V) be drawn, one of them (P V) through a focus (S), the other (P F) at right angles to the tangent, and cutting the focal axis in (I), a straight line (I G) drawn through the extremity of the normal in the axis, at right angles to the line through the focus, will cut off from it a segment (P G) equal to half the principal parameter. APPENDIX. PROP. I. Fig. 95, 96. 97.

First, let the section be an ellipse or hyperbola. Let AB be the transverse, and DE the conjugate axis, C being the center. Let HL be the diameter parallel to the tangent, and let it meet PS in K, and let SY be perpendicular to the tangent. Then, as SY, FP are at right angles to the tangent, they are parallel; and therefore (29. i.) the angles YSP, GPI are equal, and the angles at G and Y are equal, each of them being a a right one. Hence, Fig. 95, 96.

(4. vi.) SY : SP : : PG : PI : : PF : PK, or its equal CA, by article 108. Consequently,

$PG \times CA = PI \times PF = CD^2$, by article 109; for PF is equal to a straight line drawn from C to the tangent, and at right angles to it. It therefore follows that

APPEN-DIX. CA : CD :: CD : PG; and therefore, by article 72, PG is equal to half the parameter of AB.

Fig. 97. Secondly, let the section be a parabola, and let the tangent meet the axis AI in E, and let SY be perpendicular to the tangent. Then it is evident, from article 168, that IP=2SY, and the triangles YSP, GPI being equiangular,

SY : PS :: GP : PI or 2SY. But by article 160,

AS : SY :: SY : SP, and therefore $SP=\frac{SY^2}{AS}$.

Consequently, $SY : \frac{SY^2}{AS} :: PG : 2SY$, and $2SY^2=\frac{SY^2\times PG}{AS}$, and therefore PG=2AS=half the parameter of the axis, by article 146.

232. If from a point P in which a straight line PY touches a conic section, a straight line as PV be drawn through a focus, and a segment PG be taken in it equal to half the principal parameter, a perpendicular GI to PV and a perpendicular PF to PY will meet in the same point in the focal axis. This is evident from the last article.

233. In each of the three sections, the cube of the normal PI, divided by the radius of the circle of curvature, is equal to the square of half the principal parameter.

For, by article 230, in the ellipse and hyperbola, the radius of curvature is equal to $\frac{CA^2\times PI^3}{CD^4}$, and PI^3 being divided by this value of the radius, the quotient is $\frac{CD^4}{CA^2}$, which is the square of $\frac{CD^2}{CA}=PG$, by article 231.

In the parabola, by article 230, the radius of curvature is equal to $\frac{4\,PI^3}{L^2}$, and PI^3 being divided by this value of the radius, the quotient is $\frac{L^2}{4}$. APPENDIX.

234. The rest remaining, as in the last three articles, let Q be a point in the curve indefinitely near to P, and let Q R be drawn parallel to P S and meet the tangent in R, and let P V be the chord of curvature passing through S. Then, $PV = \frac{QP^2}{QR}$; for, by article 129, if P Y touch a circle in P, and the circumference pass through Q, the chord of the circle passing through S will be equal to $\frac{QP^2}{QR}$, considering Q P as a straight line. Supposing, therefore, the point Q to move up to P, and Q R always to be parallel to P S, the circle passing through P, Q and touching P Y, will be the circle of curvature, and V will be the point in which P S produced again meets the circle. PROP. II. Fig. 95, 96, 97.

235. The rest remaining as in the last article, let Q T be perpendicular to P S, and then, when Q moves up to P, $\frac{QT^2}{QR}$ will be equal to the principal parameter L. PROP. III.

For, Q being indefinitely near to P, the angle Q P T is the same with the angle Y P S, and therefore (4. and 22. vi.) $PS^2 : SY^2 :: QP^2 : QT^2 :: \frac{QP^2}{QR} : \frac{QT^2}{QR}$; and by the last article $\frac{QP^2}{QR} = PV$. But, in the ellipse and hyperbola, by article 227, $PV = \frac{2\,CL^2}{CA}$, and therefore, in these two sections,

$$PS^2 : SY^2 :: \frac{2\,CL^2}{CA} : \frac{QT^2}{QR}.$$ Again, by article 111, (and 22. vi.) $PS^2 : SY^2 :: CL^2 : CD^2$, and therefore

APPEN-DIX. (11. v.) $CL^2 : CD^2 :: \frac{2CL^2}{CA} : \frac{QT^2}{QR}$. Consequently, $\frac{QT^2}{QR} = \frac{2CD^2}{CA} = L$.

In the parabola, by article 168, $PV = 4PS = \frac{QP^2}{QR}$, and therefore, by the above, $PS^2 : SY^2 :: 4PS : \frac{QT^2}{QR}$. Hence $PS : SY^2 :: 4 : \frac{QT^2}{QR}$; but, by article 160,

$$PS = \frac{4SY^2}{L}, \text{ and therefore,}$$

$$\frac{4SY^2}{L} : SY^2 :: 4 : \frac{QT^2}{QR} = L.$$

Fig. 98. 236. *Lemma.* If from the point A in the straight line CB, a point P move from rest at A, and proceed towards B with a velocity uniformly accelerated, and if at the commencement of this motion a point D begin to move from A towards C, and proceed with a velocity equally retarded as that of P is accelerated, the whole space PD will be generated with a uniform velocity, till D arrives at a state of rest.

For the sum of the velocities of P and D will be the same at every instant during the description of the space PD, as the decrease in the velocity of D is made up by an equal increase in the velocity of P.

237. If B, C be the points at which P, D arrive when the velocity, and consequently the motion of D, is entirely destroyed, the spaces AB, AC are equal.

For if D now begin to move from rest at C, and proceed towards A with a velocity exactly the converse of that with which it moved from A to C, it will move back to A with this converse velocity in the same time that it before moved from A to C; and it is evident that this converse velocity of D, and that with which P moved from A to B, will always be equal at equal distances from C and A. The spaces A B, A C will therefore be equal. APPENDIX.

238. If a body move from rest, and proceed in a straight line with a velocity uniformly accelerated, the space passed over in a given time will be equal to half the space it would pass over, in the same time, with the last acquired velocity continued uniform.

This is evident from the last two articles. For the acquired velocity of P when it arrives at B is that with which the space C B is uniformly described in the time that P moves from A to B; and by the last article A B is half C B.

239. *Lemma.* If S, s denote two spaces passed over by bodies moved from rest, and proceeding by the action of uniform forces with velocities uniformly accelerated, and if T, t denote the times during which they continue to move, and V, v represent the velocities acquired, then, by the last article and Mechanics,

$2S : 2s :: TV : tv$,

or $S : s :: TV : tv$. But again, by Mechanics,

$V : v :: F \times T : f \times t$, and therefore

$S : s :: F \times T^2 : f \times t^2$.

Consequently $F : f :: \frac{S}{T^2} : \frac{s}{t^2}$, or the force is as the space moved over divided by the square of the time.

The preceding articles enable us to investigate the law under which a force attracting to a fixed center, combined with one acting in the direction of a tangent, causes a body to move in the curve of a conic section. The three articles immediately following are here inserted, on a supposition that they may illustrate some propositions in the second and third Sections of the Principia.

240. Let the body P revolve in the ellipse A P B, and let the law of centripetal force tending to C the center Fig. 99.

APPENDIX. be required. Let A B be the transverse axis, and C D the semiconjugate; let P R touch the ellipse, and let the diameter L H be parallel to it, and consequently the conjugate diameter to P C; let P F be perpendicular to the tangent, and therefore (29. i.) at right angles to L H; let Q be a point in the curve indefinitely near to P, and let Q R, parallel to P C, be drawn to the tangent. Let P V, in P C produced, be the chord of curvature for the point P. Then in this case Q R is equal to S in the last article, being the space over which the central force causes a body to deviate from the tangent; and $CP^2 \times QT^2 =$ the square of the time; consequently, the centripetal force $= \frac{QR}{CP^2 \times QT^2}$. But as L H, P R are parallel, the angles R P T, or, as Q is indefinitely near to P, Q P T, P C F, (29. i.) are equal; and as the angles at F and T are right ones (4. and 22. vi.)

$$CP^2 : PF^2 :: PQ^2 : QT^2,$$

and $CP^2 \times QT^2 = PF^2 \times PQ^2$. Again, as in articles 234, 235, $PV = \frac{PQ^2}{QR} = \frac{2CL^2}{CP}$, and therefore $QR = \frac{PQ^2 \times CP}{2CL}$. The centripetal force is therefore, by substitution, $= \frac{PQ^2 \times CP}{2CL^2} \times \frac{1}{PF^2 \times PQ^2}$ $= \frac{CP}{2CL^2 \times PF^2} = \frac{CP}{2CA^2 \times CD^2}$ by article 112. Consequently, as $\frac{1}{2CA^2 \times CD^2}$ is a constant quantity, the centripetal force is as the distance of the revolving body from the center.

Fig. 95, 96. 241. Let a body P move in an ellipse or hyperbola A P, and let the centripetal force tending to the focus S of the section be required. Let A B be the transverse, and D E the conjugate axis. Let P Y touch the section, and let L H be the diameter parallel to it. Let S Y be perpendicular to the tangent, and let P S passing through the focus meet H L in K, and let the segment P V be the chord of curvature. Let Q be a point in the curve indefinitely near to P, and let Q T be perpendicular to S P, and let Q R parallel to S P be drawn to the tangent. Then, for the same reasons as in the last article, the centripetal force $= \frac{QR}{SP^2 \times QT^2}$. But, by article 235, $\frac{QT^2}{QR} = L$ the principal parameter, and therefore the centripetal force $= \frac{1}{SP^2 \times L}$; and as L is a constant quantity, the centripetal force is inversely as the square of the distance of the revolving body from the focus. APPENDIX.

242. Let a body P move in the curve of a parabola A P, and let the law of centripetal force tending to the focus S be required. Let P S be drawn through the focus, and Q being a point indefinitely near to P, let Q T be drawn to it at right angles, and let Q R be parallel to S P. Then, for the same reasons as in article 240, the centripetal force $= \frac{QR}{SP^2 \times QT^2}$. But, by article 235, $\frac{QT^2}{QR} = L$ the principal parameter, and therefore the centripetal force $= \frac{1}{SP^2 \times T^2}$; and as L is a constant quantity, the centripetal force is inversely as the square of the distance of the revolving body from the focus. Fig. 97.

243. If from the vertex (A) of the diameter (A B) of a parabola, a straight line (A D) be drawn to the extremity (D) of an ordinate (D B) to the diameter, and PROP. IV. Fig. 100.

APPENDIX. a diameter (K F) be drawn bisecting it (in I,) the abscis (F I) of this last mentioned diameter will be one fourth of (A B) the first mentioned absciss.

For let a tangent D E meet the diameter A B in E, and the diameter K F in C, and then, as by article 121, E A, C I are parallel, and as D I is half of D A, C I is half of E A, or, by article 142, equal to half A B. But, by article 142, F I is half of C I, and therefore equal to one fourth of A B.

244. Two conic sections, or two segments of conic sections, are called *similar segments*, if a rectilineal figure can be inscribed in one of them similar and similarly situated to a rectilineal figure inscribed in the other.

PROP. V. 245. Similar parabolic segments may be taken in any two parabolas, and the similar rectilineal figures inscribed in them will be to one another as the squares of the principal parameters.

Fig. 100, 101. For, let D A H, *d a h* be any two parabolas of which A B, *a b* are the axes; and let P, *p* be the principal parameters.

Let P be to the absciss A B, as *p* to the absciss *a b*, and let D B H, *d b h* be the corresponding double ordinates; and then, as by article 136, $P \times AB = BD^2$, and $p \times ab = bd^2$,

$P \times AB : p \times ab :: BD^2 : bd^2$, and therefore (22. vi.)

$AB : ab :: BD : bd$, and alternately

$AB : BD :: ab : bd$; and therefore (6. vi.) the triangles A B D, *a b d* are similar. In the same way it may be proved that the triangles A B H, *a b h* are similar, and therefore the whole triangle A D H is similar to the whole triangle *a d h*; and it is evident that the homologous sides A D, *a d* are to one another as P to *p*. Again, let the diameters F K, *f k* bisect A D, *a d* respectively in I, *i*, and meet the curve in F, *f* respectively, and let A F, F D, *af*, *fd* be drawn, and then, by article 243, F I is equal to one fourth of A B, and *f i* is equal to one fourth of *a b*. Consequently, (11. v.) A I : I F :: *a i* : *i f*, and (29, i.) as the angles A I F, D A B, *a i f*, *d a b* are equal, the triangles A I F, *a i f* are similar. For the same reasons the triangles F I D, *f i d* are similar; and in the same way it may be proved, that if diameters bisect A H, *ah*, and meet the curve respectively in G, *g*, and the triangles A G H, *a g h* be completed, they will be similar. The whole rectilineal figure, therefore, D F A G H D, is similar to the whole rectilineal figures *d f a g h d*, and they are to one another as the squares of their homologous sides, or, as is evident from the above, as the squares of the parameters P, *p*. APPENDIX.

Similar segments may also be cut off by double ordinates to any diameters, provided the ordinate and its diameter in the one parabola, and the ordinate and its diameter in the other, contain equal angles, and the corresponding abscissas be to one another as the parameters of these diameters. This may be proved by proceeding as above.

246. In two parabolas, the parameters of diameters which contain equal angles with their ordinates, are to one another as the parameters of the axes. For the rest remaining, let R be the parameter of F I, and r that of $f i$, and then, by articles 136, 243, $R \times \frac{AB}{4} = AI^2$, and $r \times \frac{ab}{4} = ai$. But by the similar triangles (22. vi.) $AI^2 : ai^2 :: BD^2$ or $P \times AB : bd^2$ or $p \times ab$. Consequently $R \times \frac{AB}{4} : r \times \frac{ab}{4} :: P \times AB : p \times ab$, and therefore $R : r :: P : p$.

247. It is evident, from article 245, that similar rectilineal figures may be inscribed in the similar parabolic

APPENDIX. segments A F D H G, $afdhg$, of which the homologous sides will be to one another as P to p, and which will be deficient from the parabolic segments, by spaces less than any given. The similar parabolic segments themselves, will therefore be to one another as P^2 to p^2.

PROP. VI. 248. Two ellipses or two hyperbolas will be similar, if the axes of the one be proportional to the axes of the other.

Fig. 105, 106. Let A H, ah be two ellipses or two hyperbolas, of which the transverse axes are A B, ab, and conjugate D E, de, and let C be their common center. Then, if $AB : DE :: ab : de$, the ellipses are similar to one another, and the hyperbolas are similar to one another.

For let the transverse axes be in the same straight line, and then the conjugate axes will fall upon one another, as represented in the figures.

Let H C L, hCl be any other two diameters in the ellipses in the same straight line, or any other two transverse diameters in the same straight line in the hyperbolas. From the vertices H, h let H G, hg be ordinates to A B, ab, and then the triangles C G H, cgh will be similar. By hypothesis

(and 22. vi.) $CA^2 : CD^2 :: Ca^2 : Cd^2$.

By article 68, $CA^2 : CD^2 :: AG \times GB : GH^2$, and

$Ca^2 : Cd^2 :: ag \times gb : gh^2$; and therefore, (11. v.) $AG \times GB : GH^2 :: ag \times gb : gh^2$.

By similar triangles (and 22. vi.)

$GH^2 : CG^2 :: gh^2 : Cg^2$, and therefore, placing for ex æquali,

$AG \times GB : GH^2 : CG^2$

$ag \times gb : gh^2 : Cg^2$, and (22. v.)

$AG \times GB : CG^2 :: ag \times gb : Cg^2$.

Consequently in the ellipses (5. ii. and 18. v.)

$CA^2 : CG^2 :: Ca^2 : Cg^2$; and in the hyperbolas, by inversion and conversion (and 6. ii.) we have the same proportion; and therefore (22. vi.) CA : CG :: C*a* : C*g*. Again, in both curves, by similar triangles, CG : CH :: C*g* : C*h*, and again placing for ex æquali,

CA : CG : CH

C*a* : C*g* : C*h*, and therefore

CA : CH :: C*a* : C*h*. Hence, the straight lines AH, *a h* being drawn, the triangles (6. vi.) ACH, *a* C *h* are similar, and CA : C*a* :: AH : *a h*. Again in the hyperbolas, CG : CA :: C*g* : C*a*, and by conversion CG : AG :: C*g* : *a g*, and therefore AG : *ag* :: CG : C*g* :: CA : C*a* :: GH : *g h*. Hence (6. vi.) the triangles AGH, *a g h* are similar. If IK, *i k* be any other diameters in the same straight line, and HI, *h i* be drawn, it may be proved in the same way, that the triangles HCI, *h* C *i* are similar, and that HI is to *h i* as CA to C*a*; and if in the hyperbolas ordinates be drawn from I, *i* to AB, *a b*, it may be proved in the same way that the rectilineal figures contained by them, the abscissas and AH, HI, *a h*, *h i* will be similar.

It is evident that this inscription of similar rectilineal figures may be carried on to any extent, and that any two homologous sides will always be as CA to C*a*.

249. It follows from the last article (and cor. 2. 20. vi.) that the similar rectilineal figures inscribed in similar ellipses, or in similar hyperbolic segments, are to one another as the squares of the transverse, or as the squares of the conjugate axes. And as a rectilineal figure may be inscribed in an ellipse, or an hyperbolic segment, which shall be deficient from the ellipse, or hyperbolic segment, by a space less than any given, similar ellipses and similar hyperbolic segments

APPEN-DIX. are to one another as the squares of the transverse, or as the squares of the conjugate axes.

250. If two ellipses, or two hyperbolas, be similar, the transverse axis in the one will be to the distance between the foci as the transverse axis in the other to the distance between the foci.

Fig. 105. For the rest remaining, let F be a focus in the one ellipse and f a focus in the other, and let DF, df be drawn. Then, by article 90, DF is equal to CA and df to Ca, and therefore by hypothesis (and 22. vi.) $DF^2 : CD^2 :: df^2 : Cd^2$, and by conversion (and 47. i.) $DF^2 : CF^2 :: df^2 : Cf^2$, and (22. vi.) DF : CF :: df : Cf. Consequently (15. v.) AB : 2 CF :: ab : 2 Cf.

Fig. 106. In the hyperbolas, let DB, db be drawn, and then, by article 91, these lines are equal to the distances between the foci and the center in their respective opposite sections. But, by hypothesis, CB : CD :: Cb : Cd, and therefore (2. vi.) BD, bd are parallel, and therefore

$$CB : BD :: Cb : bd,$$

$$\text{and } AB : 2\,BD :: ab : 2\,bd.$$

251. *Scholium.* It may be proper in this place to direct the attention of the reader to methods of ascertaining certain particulars in a conic section, supposing the curves of the sections to be given, or straight lines to be given, for the description of the curves. These methods, now to be described, might have been delivered as Corollaries to the Propositions on which they depend, or they might have been put into the form of Problems; but it appeared more adviseable to reserve them for a series of articles in a Scholium. A cautious desire against interrupting the reader in the acquisition of new truths suggested this delay. The ease with which they are deduced from preceding demonstra-

tions, and the importance of the articles themselves induced the author to think that the following was the most proper manner of delivering them, and the most proper place to insert them. APPENDIX.

252. Let the ellipse A D B E, or the opposite hyperbolas A, D B E, be given to find the center. Fig. 117. 118.

In the ellipse and in either hyperbola draw the two parallel straight lines D E, F G, and draw A B bisecting D E in H and F G in K. Let A B meet the curve of the ellipse, or the curves of the opposite hyperbolas, in A and B. Bisect A B in C, and C will be the center, by articles 61, 63.

If only one hyperbola D B E be given, two other straight lines must be drawn parallel to one another, but not parallel to D E, F G, and a straight line being drawn bisecting them will be a diameter. Its concourse therefore with A B will determine the center. Fig. 118.

253. The curve of a conic section and a point in it being given, let it be required to draw a diameter through the given point.

If the section be an ellipse, or hyperbola, find the center by the preceding article, and through the center and the given point draw a diameter. If the section be a parabola, find a diameter, by article 134, and parallel to it draw a straight line through the given point; and, by article 121, this will be the diameter required.

254. The curve and a diameter A B of a conic section being given, and any point G in the curve besides a vertex of the given diameter, let it be required to draw a straight line from G ordinately applied to the diameter.

First, let the section A G B D be an ellipse. Find the center C by article 252, and through it draw the diameter G L. Through the vertex L draw the straight line L F parallel to A B. Then if L F touch the ellipse, A B, L G will be conjugate diameters, by Fig. 117.

APPENDIX. with the part Y E of the moving ruler, and let one end of it be fixed to the ruler at E, and let the other end be fixed to the point F. By means of the pin P let the thread or string be stretched, and the part between P and E be kept close to the edge of the ruler. While the ruler G Y E slides along the edge D X of the fixed ruler, and the thread or string is kept uniformly tense, let the point of the pin P trace the line A P B C on the plane, in which the line D X and the point F are situated. The line A P B C will be the curve of a parabola, of which D X is the directrix and F the focus, as is evident from article 157.

Several writers on Conic Sections have defined the ellipse, hyperbola, and parabola by the description in articles 256, 257, 258 respectively; and from these descriptions, as founded on a primary, they have deduced other properties of the sections.

Fig. 114, 115. 259. Two straight lines A B, D E being given, bisecting one another in C but not at right angles, let it be required to describe an ellipse, or hyperbola, of which A B, D E shall be conjugate diameters, and C the center.

In C D, produced in the ellipse but between C and D in the hyperbola, take the point N, so that the rectangle under C D, D N may be equal to the square of C B. Through D draw the straight line M Q parallel to A B, and bisect C N in I. Draw I P perpendicular to C N, and let it meet M Q in P; and then it is evident (4. i.) that straight lines drawn from P to N and C will be equal. With P as a center therefore, and P N or P C as a distance, let the circle M C Q N be described, and let it meet the straight line M Q in M and Q. Draw the straight lines Q C, M C; and from D draw D K perpendicular to Q C, and D H perpendicular to M C. In Q C take C L, C R each a mean proportional between C Q, C K; and in M C take C F and C G each a mean proportional between C M, C H. Then will R L, G F be the axes of the ellipse, or hyperbola, proposed to be described, as is evident from article 85, (and 31. iii.) and article 78. Consequently the foci may be found, and the descriptions of the curves may be completed, as in articles 256, 257. APPENDIX.

260. The straight line G E being given in position and magnitude, and the straight line A B bisecting it in B, let it be required to describe a parabola, of which A B shall be a diameter, and G E a double ordinate to it. Let the straight line P be a third proportional to A B, B G, and produce B A to Y, so that A Y may be a fourth part of P. Through Y draw D X at right angles to Y B, and through A draw A N parallel to G E. Make the angle N A F equal to the angle N A Y, and make A F equal to A Y. A parabola described with the focus F and the directrix D X, as in article 258, will be the section required, as is evident from articles 66, 158, 168, 169. Fig. 116.

261. The diameter A B of an ellipse or hyperbola P B, and P L an ordinate to it being given, let it be required to find the diameter conjugate to A B. Fig. 111, 112.

Let the straight line M be a mean proportional (13. vi.) between the abscisses A L, L B; and, C being the center of the section, let M be to P L as C B to C D a straight line parallel to P L. Then will C D be half the conjugate diameter required. For, by hypothesis, (and 22. vi.) $M^2 : PL^2 :: CB^2 : CD^2$. But (17. vi.) $M^2 = AL \times LB$, and therefore $AL \times LB : PL^2 :: CB^2 : CD^2$. Consequently C D is the semiconjugate diameter to A B, by articles 63, 68.

262. Let E D be an equilateral hyperbola, of which A F, A C are the asymptotes, and let it cut in the point PROP. VII. Fig. 102.

APPEN-DIX. D the curve of the parabola A D, of which A F is the axis, and the segment A F equal to the parameter of the axis; let there be drawn to the curve of the hyperbola the straight line F E parallel to the asymptote A C, and from the point D, in which the curves of the hyperbola and parabola cut one another, let there be drawn to the asymptote A F the straight line D B parallel to the asymptote A C; then will the straight lines B D, A B be two mean proportionals between A F, F E.

For as E D is an equilateral hyperbola, the angle A F E is a right one, by article 183, (and 29. i.). The straight line D B is therefore an ordinate to A B the axis of the parabola, and, by article 136,
$AF \times AB = BD^2$. In the hyperbola, by article 184.
$AB \times BD = AF \times FE$. By the first of these equations,

$AF : BD :: BD : AB$; and by the second,

$AF : BD :: AB : FE$. Hence,

(11. 5.) $AF : BD :: BD : AB :: AB : FE$.

263. Hence if two straight lines as A F, F E be given, two mean proportionals may be found between them. For let the two straight lines A F, F E be at right angles to one another, and let the parallelogram A F E G be completed. Let the parabola A D be described, of which A F is the axis, and the segment A F equal to its parameter. Again, let an equilateral hyperbola be described through the point E, of which A F, A G are the asymptotes, and let its curve cut the curve of the parabola in D. Let D B be drawn to A F and parallel to A G, and let D C be drawn to A G parallel to A F. Then the straight lines B D, A B will be two mean proportionals between A F, F E.

PROP. VIII. Fig. 103. 264. Let A E be a parabola, of which A D is the axis, and A B a segment in it equal to half its parameter; let the straight line B G be perpendicular to the axis, and draw A G; with the center G and distance G A describe the circle A C E cutting the axis in the point C and the curve of the parabola in E, and let E D be drawn an ordinate to the axis; the straight lines E D, A D will be two mean proportionals between A C and a straight line equal to the double of G B. APPENDIX.

For let D E meet the circle in F, and in D E produced let E H be taken equal to D F. By the construction (and 3. iii.) A C is equal to the parameter of the axis, and therefore, by article 138, $DE^2 = DA \times AC$. Again (36. iii.) $DE \times DF = DA \times DC$, and therefore $DE^2 + DE \times DF = DA \times DC + DA \times AC$. But $DE^2 + DE \times DF = DE \times (DE + DF) = DE \times DH$; and (2 ii.) $DA \times AC + DA \times DC = AD^2$. Hence DE : AD :: AD : DH, and as AC : DE :: DE : AD, we have AC : DE :: DE : AD :: AD : DH. But D H is double of G B; for let G I be drawn parallel to A D, and let it meet D H in I. Then G B, I D (34. i.) are equal to one another, as are also (3. iii.) E I, I F to one another, and therefore H I is equal to I D. The Proposition is therefore evident.

265. Hence, by means of a parabola and a circle, a method is evident of finding two mean proportionals between two given straight lines.

266. From any point B in the curve of the equilateral hyperbola B E let the straight lines B A, B D be drawn to the asymptotes C A, C D, and let B A be parallel to C D and B D parallel to C A, and let A D the diameter of the parallelogram be drawn. With the center B and a distance equal to the double of A D let a circle be described, and let it meet the curve of the hyperbola in E. From E draw E F to the asymptote C D and parallel to C A; and then, A F being drawn, the angle B A F will be a third part of the angle B A D. PROP. IX. Fig. 104.

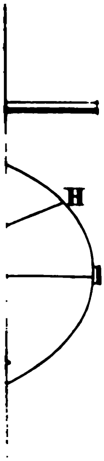
For let A F meet B D in G. Bisect D F in K, and

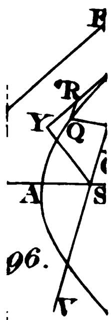
APPENDIX. draw K I parallel to B D, and let it meet A F in the point I. Draw D I. Then, as the hyperbola is equilateral, the angle A C D is a right one, and therefore (29. i.) each of the angles F K I, D K I is a right one, and (4. i.) F I, I D are equal. But, on account of the equals F K, K D and the parallels K I, D G, F I is equal to I G. Again, (15. and 29. i.) the triangles A B G, F C A are equiangular, and therefore (4. vi.) A B : B G : : C F : C A, and (16. vi.) B G × C F = A B × C A = (34. i.) C D × D B = C F × F E, by article 184, and B G = F E. Consequently (33. i.) B E is equal to G F; and therefore, by the construction, F I, I D, D A are equal. The angles D F I, F D I are therefore equal to one another, as are also the angles D I A, D A I to one another. The angle D A I is therefore equal to the double (32. i.) of D F I, or of its equal the angle B A G. Consequently the angle B A G is equal to a third part of the angle B A D.

267. Hence, by means of an equilateral hyperbola and its asymptotes, an angle may be divided into three equal parts.

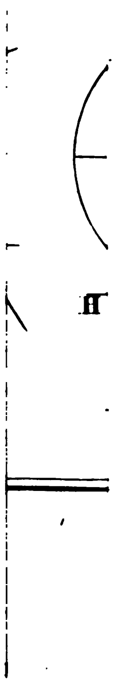
PROP. X. 268. If a scalene cone be cut through the axis by a plane perpendicular to the base, of the sides of the section, meeting in the vertex, one will be the greatest, and the other the least of all the sides of the cone.

Fig. 107. 108. For let V N O P be a scalene cone, of which V is the vertex, N O P the base, and C the center of the base. Let the straight line V B be perpendicular to the plane of the base, and meet it in B. Draw the straight line B C, and let it meet the circumference of the base in the points P, N. Through the straight lines V B, B C let a plane be passed, and let it cut the cone; and let the section formed, with the cone, be the triangle V N P, as in article 15. Then as the plane of the triangle V N P passes through C, it cuts the





96.



cone through the axis *; and, as it passes through VB, it is also (18. xi.) perpendicular to the base NOP. If therefore the point B be farther from N than from P, it remains to be proved, that VN is greater and VP less than any other side of the cone. APPENDIX.

Let VO be any other side of the cone, and let it meet the circumference of the base in O, and draw BO. Then, as VB is perpendicular to the plane of the base, the angles VBN, VBP, VBO are right angles; and therefore (47. i.) $VN^2 = VB^2 + BN^2$, $VO^2 = VB^2 + BO^2$, and $VP^2 = VB^2 + BP^2$. But (7. and 8. iii.) $BN > BO$, and $BO > BP$; and therefore $BN^2 > BO^2$, and $BO^2 > BP^2$. Consequently $VN^2 > VO^2$, and $VO^2 > VP^2$. Of all the sides of the cone therefore, VN is the greatest, and VP is the least.

If the perpendicular VB fall into the circumference of the base, then B and P will coincide; and (referring to 15. iii. instead of 7. and 8. iii.) the demonstration will be the same as above †.

269. As there can be only one perpendicular to a plane (13. xi.) drawn from the same point above it, it is evident from the demonstration of the last article, that only one plane can cut a scalene cone through the axis, and be perpendicular to the base.

270. *Lemma.* If the plane figure AFD be bounded by the straight line AD and the curve AFD, and if the square of the straight line FI, drawn from any point F in the curve perpendicular to AD, be equal to the rectangle under the segments AI, ID, the figure will be a semicircle. Fig. 109.

* As the representation of the axis could not render the demonstration more perspicuous, it was intentionally omitted in the figures.

† It is evident that all the sides of a right cone are equal to one another. For, in this case, the perpendicular to the base, drawn from V, will fall into C, the center.

APPEN-DIX. For let A D be bisected in C, and draw C F. Then (47. i.) $CF^2 = CI^2 + IF^2 = CI^2 + AI \times ID$, by hypothesis. But as A D is bisected in C, (5. ii.) $CA^2 = CD^2 = CI^2 + AI \times ID$, and therefore $CF^2 = CA^2 = CD^2$, and consequently $CF = CA = CD$. The figure A F D is therefore a semicircle.

PROP. XI. Fig. 110. 271. Let the scalene cone V N O P be cut by a plane passing through the axis, and perpendicular to the base N O P, and let the common section be the triangle V N P; in the side V P take any point D, and in the plane of the triangle make the angle V D A equal to the angle V N P; then if the cone be cut by a plane passing through D A, and perpendicular to the triangle V N P, its common section A F D B with the cone will be a circle.

For let the side V P be less than the side V N, as in article 268, and produce A D to T and N P to R. Then, as V N is greater than V P, the angle V P N (18. i.) is greater than the angle V N P. But the angles V N P, V D A are equal, by hypothesis; and as the angle P D T is equal (15. i.) to the angle V D A, the angle V P N is greater than the angle P D T. The angles V P N, D P R together are therefore greater than the angles P D T, D P R together; and consequently the angles P D T, D P R together are less than two right angles. If therefore the straight lines A D, N P be sufficiently produced they will meet. Let them be produced and meet in R; and let the plane of the section A F D B cut the plane of the base N O P in the straight line R S. In D A take any point I, and let the cone be cut by a plane passing through I and parallel to the base; and let the section formed be the circle H F K B, as in article 17. Let the circle H F K B cut the triangle V N P in the straight line H I K, and the section A F D B in the straight line F I B. Then as the section H F K B is parallel to the plane of the base, and as these parallel planes are cut by the plane of the section A F D B, the common sections (16. xi.) F I B, S R are parallel; and as the plane of the section A F D B, and the plane of the base N O P are perpendicular to the plane of the triangle V N P and cut one another in S R, the straight line S R (19. xi.) is perpendicular to the plane of the triangle V N P. The straight line F I B (8. xi.) is therefore perpendicular to the plane V N P, and consequently (4. xi.) perpendicular to H K, A D. But, as the plane of the triangle V N P passes through the axis, H K is a diameter of the circle H F K B by article 17, and therefore (3. iii.) F B is bisected in I. Consequently (35. iii.) $KI \times IH = FI^2 = BI^2$. Again, as the circle H F K B is parallel to the plane of the base, and as these parallel planes are cut by the triangle V N P, the common sections (16. xi.) H I K, N P are parallel. The angle A H I (29. i.) is therefore equal to the angle V N P, and consequently equal to the angle K D I. The angles (15. i.) A I H, K I D are also equal, and therefore the triangles A I H, K I D are equiangular. Consequently (4. vi.) $AI : IH :: KI : ID$, and (16. vi.) $KI \times IH = AI \times ID = FI^2 = BI^2$, by the above. The section A F D B is therefore a circle, by article 270. APPEN-DIX.

The circle A F D B formed in a scalene cone, in the manner mentioned in the Proposition, is called *a Subcontrary Section.*

272. If a conic section be a circle, and be not parallel to the base of the cone, it will be a subcontrary section. PROP. XII.

Let the cone V N O P be cut by a plane not parallel to the base N O P, and let the section A F D B formed by it, with the cone, be a circle; A F D B is a subcontrary section. Fig. 110.

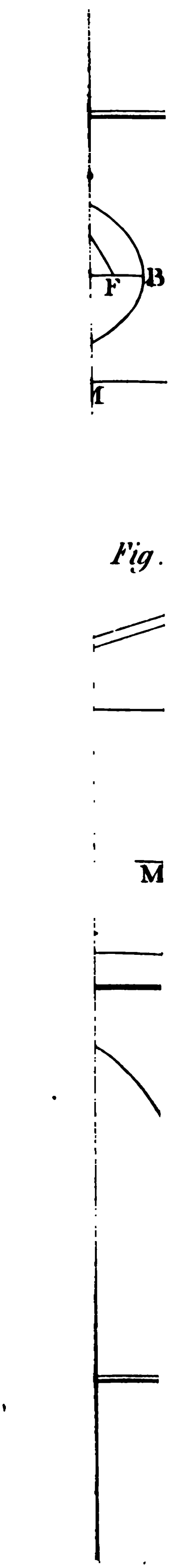
APPEN- DIX.

For let I be the point in which the axis of the cone meets the circle A F D B, and through I let a plane be passed parallel to the base, and let F K B H be the circle formed by it with the cone, as in article 17. Let B F be the common section of this circle with the circle A F D B. Then, by article 17, the point I is the center of the circle F K B H, and consequently B F is bisected in I. Through I draw in the circle A F D B the straight line A D at right angles to B F; and through A D and V, the vertex, let a plane be passed, and let V N P be the triangle formed by it with the cone, as in article 15. Let H K be the line of common section of the triangle V N P and the circle F K B H. Let L be any point in A D, and through L let a plane be passed parallel to the base N O P, or to the circle F K B H. Let M C E G be the circle formed by this plane with the cone, and let M E be its line of common section with the triangle V N P, and C L G its line of common section with the circle A F D B. Then (16. xi.) the straight lines H K, M E are parallel, as are also B I F, C L G; and as A I B is a right angle, A L C is (29. i.) right angle. Again, as the straight line A D bisects the straight line B F at right angles, A D (cor. 1. iii.) is a diameter of the circle A F D B. The straight line G C (3. iii.) is therefore bisected in L. But as I is the point in which the axis of the cone meets the circle A F D B, it is evident that the triangle V N P cuts the cone through the axis, and consequently, by article 17, M E is a diameter of the circle M C E G, and the point L is not its center. Hence the diameter M E (3. iii.) bisects G C in L at right angles, and G L is at right angles to A D, M E, and therefore it is at right angles to the triangle (4. xi.) V N P. Consequently (18. xi.) each of the sections A F D B, M G E C is at right angles to the triangle V N P, and therefore as M G E C is parallel to the base, the cone is cut by the plane V N P passing through the axis, and perpendicular to the base N O P. Again, as G L is at right angles to each of the two diameters, M E, A D, the rectangle under M L, L E is equal to the rectangle under D L, L A, each of these rectangles (35. iii.) being equal to the square of G L; and therefore (16. vi.) D L : L E : : M L : L A, and (6. vi.) the angle L D E is equal to the angle A M L, or (29. i.) V N P. The circle A F D B is therefore a subcontrary section. APPENDIX.

273. A conic section neither parallel to the base of the cone, nor a subcontrary section, is not a circle.

THE END.


F


F
B
M
Fig.
M